

A Particle Model of Our Spacetime : Origin of Gravity

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Abstract

We build a model of our spacetime by assuming new particles called “space quanta.” In the ambient or bulk spacetime $\mathcal{S}^{D_{\text{amb}}}$ ($D_{\text{amb}} \geq 4$), a multitude of space quanta form a nearly three-dimensional object, whose continuum approximation is called the space 3-brane. The world volume \mathcal{WV}_{sq} of this space 3-brane is described by an embedding $f^A(x^\mu) \in \mathcal{S}^{D_{\text{amb}}}$, which produces the induced metric $\gamma_{\mu\nu}$ on the world volume \mathcal{WV}_{sq} . This emergent spacetime $(\mathcal{WV}_{\text{sq}}, \gamma_{\mu\nu})$ from the many space quanta is proposed as the particle model of our spacetime. To study our spacetime $(\mathcal{WV}_{\text{sq}}, \gamma_{\mu\nu})$, we construct what we call the Aim-At-Target (AAT) method, which introduces an action for a 4D metric $g_{\mu\nu}$. This metric action from the AAT method can lead to General Relativity at low enough energies. The spacetime $(\mathcal{S}_{\text{GR}}, \mathbf{g}_{\mu\nu})$ of General Relativity is, at least, a good approximation to the exact or true spacetime $(\mathcal{WV}_{\text{sq}}, \gamma_{\mu\nu})$ of our universe.

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1 Introduction

The gravitational physics has been successfully understood in terms of General Relativity [1, 2, 3]. However, since the non-gravitational physics has been accurately explained by the principles of quantum mechanics, it seems necessary that General Relativity is merged with quantum mechanics [4]. For the quantum theory of gravity [4], there have been attempts such as string theory [5].

In this paper, we present a particle model of our spacetime, and explain the origin of gravity (i.e., General Relativity), as follows: in the ambient or bulk spacetime $\mathcal{S}^{D_{\text{amb}}}$ ($D_{\text{amb}} \geq 4$), there exist new particles called “space quanta.” A multitude of space quanta form a nearly three-dimensional object, which is called the “quasi-3D object.” Within this quasi-3D object, the average distance d_{sq} between nearest-neighbor space quanta satisfies $d_{\text{sq}} \lesssim O(M_{\text{P}}^{-1})$, where M_{P} is the Planck mass $\approx 10^{19}$ GeV.

At low energies $\lesssim O(0.1)d_{\text{sq}}^{-1}$, we can use a continuum approximation [6] that the quasi-3D object is replaced with a 3D continuum called the “space 3-brane.” Like a bosonic string [5, 7], this space 3-brane sweeps out its 4D “world volume” \mathcal{WV}_{sq} in the ambient spacetime $\mathcal{S}^{D_{\text{amb}}}$. This world volume \mathcal{WV}_{sq} is described by an embedding $f^A(x^\mu) \in \mathcal{S}^{D_{\text{amb}}}$, which produces the induced metric $\gamma_{\mu\nu}$ on \mathcal{WV}_{sq} . This emergent spacetime $(\mathcal{WV}_{\text{sq}}, \gamma_{\mu\nu})$ from the many space quanta is proposed as the particle model of our spacetime. The dynamics of the embedding f^A is provided by an effective theory $S_{\text{univ}}^{(3\text{br})}[f^A, \dots] = S_{\text{emb}}^{(3\text{br})}[f^A] + \dots$, where the latter ellipsis \dots denotes the action for the matter sector (e.g., the Standard Model).

To study our spacetime $(\mathcal{WV}_{\text{sq}}, \gamma_{\mu\nu})$, we construct the “Aim-At-Target (AAT) method,” which introduces an action for a 4D metric $g_{\mu\nu}$, namely, $S_{\text{univ}}^{(\text{ovlp})}[g_{\mu\nu}, \dots] = S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}] + \dots$, where the latter ellipsis \dots denotes the action for the matter sector. This new metric $g_{\mu\nu}$ is used for finding the embedding $f^A(x^\mu)$ through the equality $g_{\mu\nu} = \gamma_{\mu\nu}$, as follows:

For an easy understanding of the AAT method, we consider a simple situation that the universe contains only the space-quantum sector (i.e., the space 3-brane)—the absence of the matter sector. In this situation, we only have to study the two simpler actions $S_{\text{emb}}^{(3\text{br})}[f^A]$ and $S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}]$, which are the actions without the matter sector. For example, when the ambient spacetime $\mathcal{S}^{D_{\text{amb}}}$ is the Minkowski spacetime $\mathbb{M}^{D_{\text{amb}}}$ of the flat metric η_{AB}^{bulk} , the induced metric $\gamma_{\mu\nu}$ is represented as

$$\gamma_{\mu\nu} = \partial_\mu f_{\text{sol}}^A \partial_\nu f_{\text{sol}}^B \eta_{AB}^{\text{bulk}}, \quad (1.1)$$

where f_{sol}^A is a solution for the f^A equation of motion $\delta S_{\text{emb}}^{(3\text{br})}/\delta f^A = 0$.

For the above metric action $S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}]$, when a solution $g_{\mu\nu}^{\text{sol}}$ of $\delta S_{\text{met}}^{(\text{ovlp})}/\delta g_{\mu\nu} = 0$ satisfies the equality

$$g_{\mu\nu}^{\text{sol}} = \gamma_{\mu\nu}, \quad (1.2)$$

Eqs. (1.1) and (1.2) require that the solution f_{sol}^A of $\delta S_{\text{emb}}^{(3\text{br})}/\delta f^A = 0$ should also be a solution of the partial differential equation (PDE) for f^A

$$\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}} = g_{\mu\nu}^{\text{sol}}. \quad (1.3)$$

In other words, when the new metric $g_{\mu\nu}^{\text{sol}}$ satisfies $g_{\mu\nu}^{\text{sol}} = \gamma_{\mu\nu}$, the solution f_{sol}^A of the equation of motion $\delta S_{\text{emb}}^{(3\text{br})}/\delta f^A = 0$ can be found by solving the PDE $\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}} = g_{\mu\nu}^{\text{sol}}$. Note that the embedding f^A is similar to the locally inertial coordinates $\xi^{\hat{\alpha}}$ of General Relativity, because the PDE $\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}} = g_{\mu\nu}^{\text{sol}}$ is similar in form to $\partial_\mu \xi^{\hat{\alpha}} \partial_\nu \xi^{\hat{\beta}} \eta_{\hat{\alpha}\hat{\beta}} = \mathbf{g}_{\mu\nu}$, where $\partial_\mu \xi^{\hat{\alpha}}$ is the vierbein [2].

To sum up, the AAT method using the metric action $S_{\text{met}}[g_{\mu\nu}]$ consists of two main steps: (i) finding a solution $g_{\mu\nu}^{\text{sol}}$ of $(\delta S_{\text{met}}/\delta g_{\mu\nu})[g_{\mu\nu}] = 0$, and next (ii) finding a solution $f_{\text{PDE-sol}}^A$ of $\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}} = g_{\mu\nu}^{\text{sol}}$. In case of $g_{\mu\nu}^{\text{sol}} = \gamma_{\mu\nu}$ ($= \partial_\mu f_{\text{sol}}^A \partial_\nu f_{\text{sol}}^B \eta_{AB}^{\text{bulk}}$), we can find a solution $f_{\text{PDE-sol}}^A$ satisfying $f_{\text{PDE-sol}}^A = f_{\text{sol}}^A$, where f_{sol}^A is what we really want to know.

Then, as far as the equality $g_{\mu\nu}^{\text{sol}} = \gamma_{\mu\nu}$ remains true, the “combination” of

$$(a) \text{ the metric action } S_{\text{met}}[g_{\mu\nu}] \quad \text{and} \quad (b) \text{ the PDE } \partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}} = g_{\mu\nu}^{\text{sol}} \quad (1.4)$$

can be used instead of the 3-brane action $S_{\text{emb}}^{(3\text{br})}[f^A]$. This is the essential feature of the AAT method.

At low enough energies, the metric action $S_{\text{met}}[g_{\mu\nu}]$ in Eq. (1.4) can be well approximated by the Einstein-Hilbert action S_{EH} of General Relativity. Then, this Einstein-Hilbert action S_{EH} can be a good low-energy description for the 3-brane action $S_{\text{emb}}^{(3\text{br})}[f^A]$ in the absence of the matter sector. When the matter sector is present, the whole action of General Relativity can be a good low-energy description for the above “universe action” $S_{\text{univ}}^{(3\text{br})}[f^A, \dots]$. In this manner, the AAT method can produce General Relativity at low enough energies—this explains the origin of gravity (i.e., General Relativity).

Since, in the AAT method, General Relativity can be subsidiary to the universe action $S_{\text{univ}}^{(3\text{br})}[f^A, \dots]$ (see around Eq. (1.4)), we must not forget that, at the most fundamental level, our universe should be studied in terms of *physical laws within the ambient spacetime* $\mathcal{S}^{D_{\text{amb}}}$ which govern both the space-quantum and matter sectors of our universe. These physical laws within the ambient spacetime $\mathcal{S}^{D_{\text{amb}}}$ can be represented as quantum field theories defined on $\mathcal{S}^{D_{\text{amb}}}$ —this may shed some light on the quantum theory of gravity [4].

Meanwhile, like usual many-particle systems (e.g., superconductors), our universe as a system in the ambient spacetime $\mathcal{S}^{D_{\text{amb}}}$ consists of an enormous number of particles such as space quanta. Thus, useful ideas for the study of our universe in $\mathcal{S}^{D_{\text{amb}}}$ can be found by surveying *physics in our spacetime* \mathcal{WV}_{sq} , for example, condensed matter physics [8].

The rest of this paper is organized, as follows: in Sec. 2, the wave-particle duality of quantum mechanics is applied to the gravitational field. Since the particle nature of the gravitational field implies the existence of the new particle (i.e., space quantum), the spacetime manifold \mathcal{S}_{GR} of General Relativity is assumed to consist of (very many) space quanta—this is called the space-quantum hypothesis.

In Sec. 3, to maintain the wave nature of a single space quantum (even without any other space quanta), we assume that there exists the ambient spacetime $\mathcal{S}^{D_{\text{amb}}}$, which surrounds the spacetime \mathcal{S}_{GR} of General Relativity. To explain the 3D space part of the GR spacetime \mathcal{S}_{GR} , we assume that space quanta in $\mathcal{S}^{D_{\text{amb}}}$ form the quasi-3D object, whose continuum limit is the space 3-brane.

In Sec. 4, we deal with the kinematics of the space 3-brane, whose world volume \mathcal{WV}_{sq} is described by an embedding $f^A(x^\mu)$. The induced metric $\gamma_{\mu\nu}$ on the world volume \mathcal{WV}_{sq} can be approximated by the GR metric $\mathbf{g}_{\mu\nu}$. For simplicity, we consider the effective theory $S_{\text{emb}}^{(3\text{br})}[f^A]$ *only* for the space 3-brane (the action for matter will be studied in Sec. 7).

In Sec. 5, we present the Aim-At-Target (AAT) method for studying the effective theory $S_{\text{emb}}^{(3\text{br})}[f^A]$ of the space 3-brane. This AAT method using a metric action $S_{\text{met}}[g_{\mu\nu}]$ contains the coupled equations $\delta S_{\text{met}}/\delta g_{\mu\nu} = 0$ and $\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}} = g_{\mu\nu}$. As far as $g_{\mu\nu} = \gamma_{\mu\nu}$ holds good, the metric action $S_{\text{met}}[g_{\mu\nu}]$ can replace the 3-brane action $S_{\text{emb}}^{(3\text{br})}[f^A]$.

In Sec. 6, in terms of symmetries, we study the forms of the metric action $S_{\text{met}}[g_{\mu\nu}]$ used in the AAT method. The Diff(4)-invariant action $S_{\text{met}}[g_{\mu\nu}]$ can explain the Einstein-Hilbert action $S_{\text{EH}}[\mathbf{g}_{\mu\nu}]$, which is an essential part of General Relativity.

In Sec. 7, since the universe contains the matter sector, we consider the more general action $S_{\text{univ}}^{(3\text{br})}[f^A, \dots] = S_{\text{emb}}^{(3\text{br})}[f^A] + \dots$ for the inclusion of matter. By applying the AAT method similarly, we can obtain General Relativity at low enough energies.

2 Applying Quantum Mechanics to Gravity: Space as a Discrete System of Particles

Quantum mechanics explains many phenomena of nature very well. Thus, we can try to combine gravity with it (i.e., a quantum theory of gravitation). Because quantum mechanics has the wave-particle duality as its signature property, we further think about the basic concept **particle**: since the wave-particle duality has been successfully applied to ordinary sensible objects like light and matter, considering these ordinary objects (rather than graviton) is helpful in understanding the concept of particle.

An ordinary material object (e.g., a bearing ball) has a “substance” characterized by (i) stuff material (e.g., metal) and (ii) shape in space (e.g., ball or sphere), which may correspond to “matter” (*hyle* in Greek) and “form” (*eidos* or *morphe*) of the ancient Greek philosophy, respectively. Therefore, an object is called a “particle” if the shape of its substance is point-like in space while the object exists. The particle nature of material objects like electron is evident.

Because the important quantum phenomena like the photoelectric effect and the Davisson-Germer experiment have been observed by laboratory frames (e.g., O_{rest} of Fig. 1(a)) under the influence of gravity, the wave-particle duality of quantum mechanics must be observed by the rest frame O_{rest} . In addition, through the general covariance, the wave-particle duality is also observed by the freely-falling frame O_{FF} of Fig. 1(a).

In the weak-field situation of General Relativity (GR) [1, 2, 3], there exists a “nearly Lorentz (NL) coordinate system” x_{NL}^μ relative to which the metric $\mathbf{g}_{\mu\nu}$ of a slightly curved GR spacetime $\mathcal{S}_{\text{GR}}^{\text{weak}}$ has the components at every point p of the spacetime $\mathcal{S}_{\text{GR}}^{\text{weak}}$

$$\mathbf{g}_{\mu\nu} = \eta_{\mu\nu} + \mathbf{h}_{\mu\nu} \quad \text{with} \quad |\mathbf{h}_{\mu\nu}| \ll 1 \quad \text{at every } p \in \mathcal{S}_{\text{GR}}^{\text{weak}}, \quad (2.1)$$

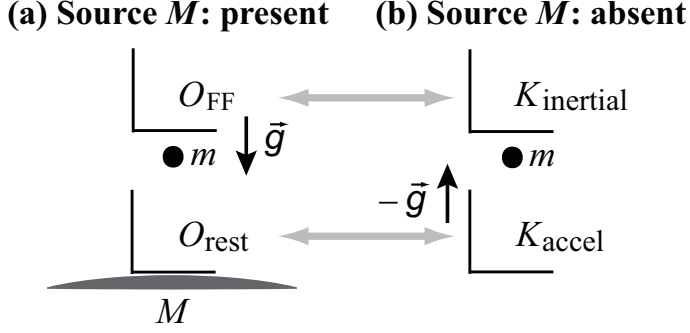


Figure 1: The correspondence between two different situations distinguished by the existence of a gravitational source M (e.g., the earth), namely, (a) the *source-present*, and (b) the *source-absent* situations. There are two kinds of “two-frame equivalences,” (i) the “non-inertial equivalence” between O_{rest} and K_{accel} , and (ii) the “inertial equivalence” between O_{FF} and K_{inertial} .

where $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$ in the *mostly plus* convention is called the flat “background metric,” and $\mathbf{h}_{\mu\nu}$ a small “perturbation” [1].

For the above NL coordinates x_{NL}^μ , the Einstein tensor $\mathbf{G}_{\mu\nu}(\mathbf{g}_{\rho\sigma}) = \mathbf{R}_{\mu\nu} - \mathbf{R}\mathbf{g}_{\mu\nu}/2$ has the series expansion in powers of the perturbation $\mathbf{h}_{\mu\nu}$ [1, 2, 3]

$$\mathbf{G}_{\mu\nu}(\eta_{\rho\sigma} + \mathbf{h}_{\rho\sigma}) = \mathbf{G}_{\mu\nu}^{(1)} + O(\mathbf{h}^2) \quad \text{with} \quad \mathbf{G}_{\mu\nu}^{(1)} \stackrel{\text{def}}{=} (\partial_\rho \partial_\nu \mathbf{h}_\mu^\rho + \dots)/2. \quad (2.2)$$

Then, the *vacuum* Einstein’s equation $\mathbf{G}_{\mu\nu} = 0$ has its first-order approximation

$$\mathbf{G}_{\mu\nu}^{(1)} = 0. \quad (2.3)$$

Under a “background Lorentz transformation” with $\frac{\partial x_{\text{NL}}^\mu}{\partial x_{\text{NL}}^\rho} \in SO(1, 3)$, the perturbation $\mathbf{h}_{\mu\nu}$ in Eq. (2.1) transforms like $\mathbf{h}'_{\mu\nu} = \frac{\partial x_{\text{NL}}^\rho}{\partial x_{\text{NL}}^\mu} \frac{\partial x_{\text{NL}}^\sigma}{\partial x_{\text{NL}}^\nu} \mathbf{h}_{\rho\sigma}$ as if it were a Lorentz tensor defined on the flat Minkowski spacetime \mathbb{M}^4 . This leads to the “flat-spacetime fiction” that the tensor $\mathbf{h}_{\mu\nu}$ belongs to a *theory in the flat spacetime* \mathbb{M}^4 [1]. This fiction is supported by the Fierz-Pauli (F-P) theory, where gravity is described by a symmetric tensor on the flat spacetime \mathbb{M}^4 [9].

Because the F-P theory shares $\mathbf{G}_{\mu\nu}^{(1)} = 0$ with General Relativity, the curved spacetime $\mathcal{S}_{\text{GR}}^{\text{weak}} = (\mathbb{R}^4, \eta_{\mu\nu} + \mathbf{h}_{\mu\nu})$ of General Relativity can be interpreted as the combination of (i) the flat spacetime $\mathbb{M}^4 = (\mathbb{R}^4, \eta_{\mu\nu})$, and (ii) the field $\mathbf{h}_{\mu\nu}$ propagating in this \mathbb{M}^4 . This interpretation about $\mathcal{S}_{\text{GR}}^{\text{weak}}$ is expressed as

$$\mathcal{S}_{\text{GR}}^{\text{weak}} \equiv \mathbb{M}^4 \oplus \mathbf{h}_{\mu\nu}. \quad (2.4)$$

Since the linearized vacuum Einstein’s equation $\mathbf{G}_{\mu\nu}^{(1)} = 0$ in Eq. (2.3) has plane-wave solutions, its solution $\mathbf{h}_{\mu\nu}$ in Eq. (2.4) can be the superposition of plane-wave solutions

$$\mathbf{h}_{\mu\nu}(x_{\text{NL}}) = \sum_{\sigma} \int d^3k \left[\mathbf{a}(\vec{k}, \sigma) e_{\mu\nu}(\vec{k}, \sigma) e^{+i k_\rho x_{\text{NL}}^\rho} + \mathbf{a}^*(\vec{k}, \sigma) e_{\mu\nu}^*(\vec{k}, \sigma) e^{-i k_\rho x_{\text{NL}}^\rho} \right], \quad (2.5)$$

where $e_{\mu\nu}(\vec{k}, \sigma)$ is a polarization tensor for wave vector \vec{k} and helicity σ [2].

As in the field quantization of the F-P theory, the amplitudes $\mathbf{a}(\vec{k}, \sigma)$ and $\mathbf{a}^*(\vec{k}, \sigma)$ in Eq. (2.5) are replaced with the annihilation and creation operators $\hat{\mathbf{a}}(\vec{k}, \sigma)$ and $\hat{\mathbf{a}}^\dagger(\vec{k}, \sigma)$ for a “particle” called the graviton—the wave-particle duality is applied to the “wave” $\mathbf{h}_{\mu\nu}$. The graviton for the field operator $\hat{\mathbf{h}}_{\mu\nu}(x_{\text{NL}})$ is a massless spin-2 particle moving in the flat spacetime \mathbb{M}^4 .

After the field quantization, the “classical field” $\mathbf{h}_{\mu\nu}$ in the expression $\mathcal{S}_{\text{GR}}^{\text{weak}} \equiv \mathbb{M}^4 \oplus \mathbf{h}_{\mu\nu}$ corresponds to “gravitons,” whose number is denoted by $N_{\text{gr}} (\geq 1)$. Then, the classical relation $\mathcal{S}_{\text{GR}}^{\text{weak}} \equiv \mathbb{M}^4 \oplus \mathbf{h}_{\mu\nu}$ in Eq. (2.4) is replaced with its semi-classical counterpart

$$\mathcal{S}_{\text{GR}}^{\text{weak}} \equiv \mathbb{M}^4 \oplus \text{gravitons} , \quad (2.6)$$

which means that the curved spacetime $\mathcal{S}_{\text{GR}}^{\text{weak}}$ of General Relativity is the combination of (i) the flat spacetime \mathbb{M}^4 and (ii) the N_{gr} gravitons moving in this \mathbb{M}^4 .

In other words, the curved GR spacetime $\mathcal{S}_{\text{GR}}^{\text{weak}}$ is formed by adding the gravitons (i.e., *particles*) to the flat spacetime \mathbb{M}^4 . The “gravitons” in Eq. (2.6) can be regarded as the building blocks of the difference between $\mathcal{S}_{\text{GR}}^{\text{weak}}$ and \mathbb{M}^4 . For example, the GR spacetime $\mathcal{S}_{\text{GR}}^{\text{weak}}$ depends on the number N_{gr} and the locations of the gravitons.

Of course, the flat spacetime \mathbb{M}^4 in Eq. (2.6) may be such a bizarre entity that it does not contain any particles unlike the curved spacetime $\mathcal{S}_{\text{GR}}^{\text{weak}}$. However, this \mathbb{M}^4 shares the same name “spacetime manifold” with the $\mathcal{S}_{\text{GR}}^{\text{weak}}$, which surely contains particles (i.e., the N_{gr} gravitons in Eq. (2.6)). Thus, the analogical reasoning based on its sharing the same name favors the opposite opinion that the \mathbb{M}^4 contains particles like the $\mathcal{S}_{\text{GR}}^{\text{weak}}$. Moreover, since the quantum theory of gravitons is possible for non-flat “background spacetimes” (e.g., an expanding universe) [10], the flat spacetime \mathbb{M}^4 cannot be the “only” background spacetime for the definition of gravitons.

Therefore, we assume that the background spacetime \mathbb{M}^4 is composed of *particles*, whose number is denoted by $N_{\text{BS}} (\geq 1)$. This implies, through “ $\mathcal{S}_{\text{GR}}^{\text{weak}} \equiv \mathbb{M}^4 \oplus \text{gravitons}$ ” in Eq. (2.6), that the “full GR spacetime” $\mathcal{S}_{\text{GR}}^{\text{weak}}$ is also composed of particles (e.g., the N_{BS} particles + the N_{gr} gravitons).

This conclusion that $\mathcal{S}_{\text{GR}}^{\text{weak}}$ consists of particles is based on the particular form of the metric $\mathbf{g}_{\mu\nu}$ in Eq. (2.1), which is unchanged only for special types of coordinate transformations among all the transformations of General Relativity [1]. Despite this, the conclusion about $\mathcal{S}_{\text{GR}}^{\text{weak}}$ can be true for all the other coordinate transformations, because our theory can produce General Relativity as a prediction (see Sec. 7).

To explain that the flat and curved spacetimes \mathbb{M}^4 and $\mathcal{S}_{\text{GR}}^{\text{weak}}$ of General Relativity are composed of particles, we make a hypothesis that

$$\text{every spacetime manifold } \mathcal{S}_{\text{GR}} \text{ of General Relativity is composed of } \textit{particles}, \quad (2.7)$$

which has the *meaning* that

every point p of the GR spacetime \mathcal{S}_{GR} has a three-dimensional (3D) spacelike neighborhood $\mathcal{N}_{\text{space}}^{3\text{D}}(p)$ which is a “continuum approximation” to a discrete system composed of particles. (2.8)

Since the concepts like substance and shape are basically defined at a constant time, the “3D spacelike neighborhood $\mathcal{N}_{\text{space}}^{3\text{D}}(p)$ ” in Eq. (2.8) can represent (partially) the *substance of the spacetime* \mathcal{S}_{GR} . For example, the substance of a Robertson-Walker spacetime $\mathcal{S}_{\text{GR}}^{(\text{RW})}$ is wholly represented by the 3D spacelike hypersurface of a constant cosmic time t , which approximately describes a discrete system composed of particles according to Eq. (2.8).

Next, we consider a question: “Is graviton a *fundamental* building block for the substance of the GR spacetime \mathcal{S}_{GR} ?” According to Eq. (2.8), the substance of the spacetime \mathcal{S}_{GR} is a discrete system like solid materials (cf. Sec. 3). Thus, for analysis, we can use an analogy that the substance of the spacetime \mathcal{S}_{GR} corresponds to a crystal composed of many lattice atoms. This atomic crystal can experience a large-scale deformation of its lattice. In the quantum-mechanical framework, the lattice deformation of longer wavelengths than the lattice spacing(s) can be analyzed by introducing a quantized normal mode called the “phonon” [8]. This bosonic quasi-particle, phonon, differs much in moving range from the lattice atom, which is confined to a small region around its equilibrium position.

If the graviton corresponds to the lattice atom in the above analogy, then (i) the graviton (e.g., a plane-wave solution moving at the speed of light) should be confined to a small region like the lattice atom, and (ii) there exist collective vibrational motions of many “lattice gravitons,” i.e., the lower-energy excitations corresponding to the phonon in the analogy. Since these two conclusions do not seem plausible, the graviton does not correspond to the lattice atom but to the phonon in the analogy.

Therefore, we formulate the “space-quantum hypothesis” that

every point p of the GR spacetime \mathcal{S}_{GR} has a 3D spacelike neighborhood $\mathcal{N}_{\text{space}}^{3\text{D}}(p)$ which is a continuum approximation to a discrete system ***Syst***_{sq} composed of particles called **space quanta**, (2.9)

which is the final meaning of the hypothesis in Eq. (2.7). Like the phonon, the graviton is an *emergent phenomenon* arising through interactions among space quanta (see Secs. 6 and 7), implying each of these space quanta is different and more fundamental than the graviton.

If the substance of every space quantum has a point-like shape, the space quantum is a particle. However, the point-like shape of the space quantum (and the other kinds of quanta) may be only an approximation based on the smallness of its substance compared with the observational precision. Then, the space quantum may be a spatially p_{br} -dimensional object such as a string ($p_{\text{br}} = 1$), or a composite system made up of two or more objects which interact weakly and/or strongly. However, in this paper, the space quantum is regarded as

a particle of point-like shape (i.e., $p_{\text{br}} = 0$), if the assumption of $p_{\text{br}} = 0$ produces General Relativity as a low-energy effective theory (see Secs. 6 and 7).

Since the space-quantum hypothesis implies the space part of the GR spacetime \mathcal{S}_{GR} is essentially a discrete object, the hypothesis is different from the basic axiom of General Relativity that spacetime is a differentiable manifold (i.e., a continuous object). This difference may not be sufficiently studied when the particle nature of gravity receives much less attention than its wave nature.

However, the discrete system of many space quanta (e.g., $\mathbf{Syst}_{\text{sq}}$) can be approximated by a 3D continuous object, when the precision of length measurement is sufficiently larger than the average distance d_{sq} between nearest-neighbor space quanta (see Sec. 3). This philosophy has been successfully used in the continuum mechanics [6].

3 The Continuum Approximation of a Space-Quantum System in the Ambient Spacetime

When space quanta forming the discrete system $\mathbf{Syst}_{\text{sq}}$ in Eq. (2.9) change their positions, the system $\mathbf{Syst}_{\text{sq}}$ undergoes a deformation. This implies, due to the space-quantum hypothesis, that the spacelike subset $\mathcal{N}_{\text{space}}^{3\text{D}}(p)$ also deforms. This deformation of the subset $\mathcal{N}_{\text{space}}^{3\text{D}}(p)$ is similarly found in General Relativity (e.g., the Schwarzschild metric) [1, 2, 3]. In addition, the deformation of the system $\mathbf{Syst}_{\text{sq}}$ can affect the motions of other objects (e.g., lights and matters) *within* the system $\mathbf{Syst}_{\text{sq}}$. This is similar to the deflection of light in General Relativity. These two similarities to General Relativity suggest the relationship between the space-quantum hypothesis and General Relativity (see Secs. 6 and 7).

When N_{sq} space quanta form a GR spacetime $\mathcal{S}_{N_{\text{sq}}}$, the motion of a *single* space quantum \mathcal{P} within $\mathcal{S}_{N_{\text{sq}}}$ can be described by its background spacetime $\mathcal{S}_{\text{bkgd}} (= \mathcal{S}_{N_{\text{sq}}-1})$, which is formed by the *other* $N_{\text{sq}} - 1$ space quanta. However, if we consider the limiting case that there are no space quanta except the single quantum \mathcal{P} (i.e., $N_{\text{sq}} = 1$), then the wave kinematics using the background spacetime $\mathcal{S}_{\text{bkgd}} (= \mathcal{S}_0)$ is not possible any longer, implying the space quantum \mathcal{P} loses its wave nature. In other words, the wave-particle duality (and thus quantum mechanics) cannot be applied to the single particle \mathcal{P} in this limiting case.

If we want to maintain the quantum mechanics (e.g., the wave nature) of the particle \mathcal{P} , a simple solution to the wave-nature problem for \mathcal{P} is to introduce another spacetime $\mathcal{S}^{D_{\text{amb}}}$ of dimension $D_{\text{amb}} (\geq 4)$ within which the space quantum \mathcal{P} moves like a particle moving within a GR spacetime \mathcal{S}_{GR} . In other words, the motion of the single space quantum \mathcal{P} is defined by the **ambient (i.e., surrounding) spacetime** $\mathcal{S}^{D_{\text{amb}}}$, without considering any other space quanta. In general, any number of space quanta can occupy the ambient spacetime $\mathcal{S}^{D_{\text{amb}}}$.

Since the spacetime \mathcal{S}_{GR} of General Relativity has the metric $\mathbf{g}_{\mu\nu}$ of the Lorentzian signature $(-, +, +, +)$, the ambient spacetime $\mathcal{S}^{D_{\text{amb}}}$ can have its own D_{amb} -dimensional Lorentzian metric g_{AB}^{bulk} ($A, B = 0, \dots, D_{\text{amb}} - 1$). For simplicity, the ambient spacetime

$\mathcal{S}^{D_{\text{amb}}}$ with the bulk metric g_{AB}^{bulk} is assumed to be the D_{amb} -dimensional Minkowski spacetime $\mathbb{M}^{D_{\text{amb}}} = (\mathbb{R}^{D_{\text{amb}}}, \eta_{AB}^{\text{bulk}})$, where the flat bulk metric η_{AB}^{bulk} is the diagonal matrix in the *mostly plus* convention

$$\eta_{AB}^{\text{bulk}} = \text{diag}(-1, +1, \dots, +1) \quad (3.1)$$

everywhere for the inertial “bulk-coordinates” Y^A ($\in \mathbb{R}^{D_{\text{amb}}}$). These bulk-coordinates Y^A are used by an inertial “bulk observer” O_{bulk} who lives in the ambient spacetime $\mathbb{M}^{D_{\text{amb}}}$.

Because it is natural that any particle performs a time evolution in every Minkowski spacetime, all space quanta occupying the ambient spacetime $\mathbb{M}^{D_{\text{amb}}}$ must execute time evolutions, producing their own world lines \mathcal{WL}_{sq} in the spacetime $\mathbb{M}^{D_{\text{amb}}}$. Here, the physics of space quantum is studied in the ambient spacetime $\mathbb{M}^{D_{\text{amb}}}$.

To explain the observation that the space part of our universe is three-dimensional, we assume that space quanta in the spacetime $\mathbb{M}^{D_{\text{amb}}}$ form a nearly 3D object, which is called the **quasi-3D object** of the many space quanta. If the average distance d_{sq} between nearest-neighbor space quanta is sufficiently smaller than the precision ΔL_{obs} of the length measurement, we can apply the continuum approximation to the quasi-3D object in the spacetime $\mathbb{M}^{D_{\text{amb}}}$, as in the continuum mechanics [6].

The “validity condition” of the continuum approximation [6] is

$$d_{\text{sq}}^3 \ll \delta V_{\text{sq}} \ll (\Delta L_{\text{obs}})^3, \quad (3.2)$$

where δV_{sq} is the volume of a 3D spacelike region $\delta \mathcal{R}_{\text{sq}}$ ($\subset \mathbb{M}^{D_{\text{amb}}}$) which contains space quanta. Since there are $\delta N_{\text{sq}} = O(\delta V_{\text{sq}}/d_{\text{sq}}^3)$ space quanta inside the region $\delta \mathcal{R}_{\text{sq}}$, the validity condition in Eq. (3.2) implies the region $\delta \mathcal{R}_{\text{sq}}$ contains many space quanta (i.e., $\delta N_{\text{sq}} \gg 1$).

We assume that the quasi-3D object satisfying the validity condition behaves like a 3D continuously-distributed system, which is called the **space 3-brane** (i.e., another name of space). In other words, the space 3-brane in the spacetime $\mathbb{M}^{D_{\text{amb}}}$ is the continuum approximation of the quasi-3D object having many space quanta, as in the continuum mechanics for solids and fluids. Mathematically, the space 3-brane composed of many space quanta is represented by a 3D spacelike submanifold of the ambient spacetime $\mathbb{M}^{D_{\text{amb}}}$.

Since the continuum approximation is applied to both of solids and fluids, we need to discuss whether the quasi-3D object (or its space 3-brane) is like a solid or a fluid: because space quanta in the fluid phase move faster, the Brownian motion can be a crucial criterion distinguishing between the two phases of the quasi-3D object. In the Brownian motion, the root-mean-square displacement Δr_{rms} of a “big” particle (e.g., proton) colliding with quick space quanta can be proportional to the square root of the elapsed time τ_{E} , namely,

$$\Delta r_{\text{rms}} \propto \tau_{\text{E}}^{1/2}. \quad (3.3)$$

This long-term behavior implies that the quasi-3D object (i.e., space) behaves like a medium which exerts random forces on the above big particle.

However, the Brownian motion caused by the fluid phase of space quanta is rejected by (i) Newton’s first law imposing $\Delta r_{\text{rms}} = (\text{initial speed}) \times \tau_{\text{E}}$ on every free particle, and (ii)

the rectilinear propagation of light in vacuum. For example, if lights from (more) distant stars were (more) deflected by the above Brownian motion, we would observe the (larger) disk-like images of the stars. In fact, the images of stars are point-like.

Therefore, the quasi-3D object of many space quanta is like a solid in our observation region. This *solid-like* quasi-3D object (a) can have a crystal lattice (e.g., simple cubic) or a non-crystalline structure like an amorphous glass, and (b) can deform elastically or plastically in response to stimuli like ordinary solid materials [8]. The deformations or strains of the quasi-3D object can be (approximately) determined by Einstein's equation $\mathbf{G}_{\mu\nu} = 8\pi G_N \mathbf{T}_{\mu\nu}$ (see Secs. 5, 6 and 7).

Since a space quantum within the solid-like quasi-3D object is not an isolated particle in the ambient spacetime $\mathbb{M}^{D_{\text{amb}}}$, the mass m_{sq} of the space quantum may differ considerably from its effective mass $m_{\text{sq}}^{(\text{eff})}$ which is affected by interactions like (i) the effective mass of electron in a solid and (ii) the constituent quark mass in a hadron.

Because space quanta in the quasi-3D object have inter-particle spacings of $O(d_{\text{sq}})$, the physical quantities of the space quanta (e.g., energy density) can vary significantly over spatial distances $\lesssim O(d_{\text{sq}})$. Thus, since the quasi-3D object resembles a discontinuously-distributed system at a high observational precision $\Delta L_{\text{obs}} \lesssim O(d_{\text{sq}})$, the above continuum approximation breaks down at the high precision $\Delta L_{\text{obs}} \lesssim O(d_{\text{sq}})$. This requires the lower bound $\Delta L_{\text{obs}}^{(c)} (\leq \Delta L_{\text{obs}})$ in order for the continuum approximation to be acceptable.

Thus, the continuum approximation considers only the larger-scale (i.e., lower-energy) behaviors of the space-quantum system, ignoring its smaller-scale (i.e., higher-energy) physics. Then, the critical precision $\Delta L_{\text{obs}}^{(c)}$ for the continuum approximation plays a similar role to the “UV cutoff” Λ_{UV} of an effective field theory [11], whose example is the Wilsonian effective action obtained by integrating out higher-energy modes than a UV cutoff.

Therefore, the continuum approximation of the quasi-3D object can be regarded as a low-energy effective theory of the space-quantum system, which has its own UV cutoff Λ_{cont} ($\sim 1/\Delta L_{\text{obs}}^{(c)}$) satisfying

$$\Lambda_{\text{cont}} = \epsilon_{\text{cont}} \times (1/d_{\text{sq}}) \quad \text{with} \quad \epsilon_{\text{cont}} \lesssim O(10^{-1}) . \quad (3.4)$$

4 The Effective Theory for the Space 3-Brane: the Bottom-Up Approach

The space 3-brane corresponds to the “continuum limit” $d_{\text{sq}} \rightarrow 0$ of the quasi-3D object which consists of many space quanta. Then, like a bosonic string [5, 7], the space 3-brane sweeps out a 4D manifold \mathcal{WV}_{sq} in the ambient spacetime $\mathbb{M}^{D_{\text{amb}}}$. The **world volume** \mathcal{WV}_{sq} of the space 3-brane is a continuum approximation to the discrete set of the world lines \mathcal{WL}_{sq} of all space quanta forming the space 3-brane.

As in the case of the 2D world sheet \mathcal{WS} of a bosonic string [5, 7], we can assume that the world volume \mathcal{WV}_{sq} of the space 3-brane is a *4D submanifold* of the ambient spacetime

$\mathbb{M}^{D_{\text{amb}}}$ (see Refs. [3, 12] for mathematical treatments): since this submanifold \mathcal{WV}_{sq} is a subset of $\mathbb{M}^{D_{\text{amb}}}$ (i.e., $\mathcal{WV}_{\text{sq}} \subset \mathbb{M}^{D_{\text{amb}}}$), there exists the inclusion map \mathbf{i} of the world volume \mathcal{WV}_{sq} , which is defined as a function

$$\mathbf{i} : \mathcal{WV}_{\text{sq}} (\subset \mathbb{M}^{D_{\text{amb}}}) \rightarrow \mathbb{M}^{D_{\text{amb}}}, \text{ satisfying} \quad (4.1)$$

$$\mathbf{i}(p) = p \in \mathbb{M}^{D_{\text{amb}}} \quad \text{for every point } p \in \mathcal{WV}_{\text{sq}}. \quad (4.2)$$

Since \mathcal{WV}_{sq} is a submanifold of $\mathbb{M}^{D_{\text{amb}}}$, the inclusion map \mathbf{i} is an immersion, i.e.,

$$\text{its derivative at } p, \mathbf{i}'_p : T_p \mathcal{WV}_{\text{sq}} \rightarrow T_p \mathbb{M}^{D_{\text{amb}}}, \text{ is injective for every } p \in \mathcal{WV}_{\text{sq}}, \quad (4.3)$$

where $T_p \mathcal{M}$ ($\mathcal{M} = \mathcal{WV}_{\text{sq}}, \mathbb{M}^{D_{\text{amb}}}$) denotes the tangent vector space of \mathcal{M} at p . In addition, the inclusion map \mathbf{i} is of constant rank 4 everywhere on \mathcal{WV}_{sq} , i.e.,

$$\text{rank}(\mathbf{i}(p)) \stackrel{\text{def}}{=} \text{rank}(\mathbf{i}'_p) = 4 \quad \text{for every } p \in \mathcal{WV}_{\text{sq}}, \quad (4.4)$$

where $\text{rank}(\mathbf{i}'_p) \stackrel{\text{def}}{=} \dim(\text{Im}(\mathbf{i}'_p))$. Then, $\text{rank}(\mathbf{i}'_p) = 4$ in Eq. (4.4) guarantees that the image $\mathbf{i}'_p(T_p \mathcal{WV}_{\text{sq}})$ of the “brane tangent space” $T_p \mathcal{WV}_{\text{sq}}$ under the map \mathbf{i}'_p in Eq. (4.3)

$$\mathbf{i}'_p(T_p \mathcal{WV}_{\text{sq}}) \stackrel{\text{def}}{=} \{ \mathbf{i}'_p(v) \text{ for } \forall v \in T_p \mathcal{WV}_{\text{sq}} \} \subset T_p \mathbb{M}^{D_{\text{amb}}} \quad (4.5)$$

is a 4D subspace of the “bulk tangent space” $T_p \mathbb{M}^{D_{\text{amb}}}$.

We are studying the submanifold \mathcal{WV}_{sq} within its ambient manifold $\mathbb{M}^{D_{\text{amb}}}$, which has a coordinate chart Y^A at every point $P \in \mathbb{M}^{D_{\text{amb}}}$: since the submanifold \mathcal{WV}_{sq} is also a manifold, the set \mathcal{WV}_{sq} as a 4D manifold has its own coordinate chart x^μ ($\mu = 0, \dots, 3$) at every point $p \in \mathcal{WV}_{\text{sq}}$. Therefore, we need to consider *two* kinds of charts at every point p of \mathcal{WV}_{sq} : (i) a “brane-chart” x^μ of \mathcal{WV}_{sq} , and (ii) a “bulk-chart” Y^A of $\mathbb{M}^{D_{\text{amb}}}$. Then, a coordinate transformation $x^\mu \rightarrow x'^\mu$ between two brane-charts x^μ and x'^μ of \mathcal{WV}_{sq} is called a “brane-to-brane (b \Rightarrow b') transformation.” Moreover, a coordinate transformation $Y^A \rightarrow Y'^A$ between two bulk-charts Y^A and Y'^A of $\mathbb{M}^{D_{\text{amb}}}$ is called a “bulk-to-bulk (B \Rightarrow B') transformation.” None of these coordinate transformations $x^\mu \rightarrow x'^\mu$ and $Y^A \rightarrow Y'^A$ change the point p of \mathcal{WV}_{sq} at all—this passive-viewpoint property is shared by any coordinate transformation between two charts in differential geometry.

Relative to the brane-chart x^μ of \mathcal{WV}_{sq} , and the bulk-chart Y^A of $\mathbb{M}^{D_{\text{amb}}}$, the equality $p = \mathbf{i}(p)$ in Eq. (4.2) has its coordinate representation

$$Y^A(p) = (Y^A \circ \mathbf{i} \circ (x^\mu)^{-1})(x^\mu(p)), \quad (4.6)$$

where $x^\mu(p) \in \mathbb{R}^4$ and $Y^A(p) \in \mathbb{R}^{D_{\text{amb}}}$.

Then, for the x^μ -and- Y^A coordinate representation of \mathbf{i} (see Eq. (4.6))

$$f^A \stackrel{\text{def}}{=} Y^A \circ \mathbf{i} \circ (x^\mu)^{-1}, \quad (4.7)$$

Eq. (4.6) defines a new kind of transformation $x^\mu \rightarrow Y^A$, called the “brane-to-bulk (b \Rightarrow B) transformation,”

$$Y^A = f^A(x^\mu) \stackrel{\text{def}}{=} f^A \circ x^\mu \quad \text{at the point } p \text{ of } \mathcal{WV}_{\text{sq}}. \quad (4.8)$$

Through the representation $f^A = Y^A \circ \mathbf{i} \circ (x^\mu)^{-1}$ in Eq. (4.7), $\text{rank}(\mathbf{i}'_p)$ in Eq. (4.4) has its x^μ -and- Y^A coordinate representation

$$\text{rank}(\mathbf{i}'_p) = \text{rank}(\partial_\mu f^A)|_{\text{at } x^\nu(p)}, \quad (4.9)$$

where the $D_{\text{amb}} \times 4$ matrix $\partial_\mu f^A$ is the Jacobian matrix of the above b \Rightarrow B transformation $Y^A = f^A(x^\mu)$.

By using the metric bulk-tensor η^{bulk} of the ambient manifold $\mathbb{M}^{D_{\text{amb}}}$, the *pullback map* \mathbf{i}^* of the inclusion map \mathbf{i} in Eqs. (4.1) and (4.2) induces a symmetric tensor $\gamma_{\mu\nu}$ on the submanifold \mathcal{WV}_{sq} in the “brane-coordinates” x^μ

$$\gamma_{\mu\nu} \stackrel{\text{def}}{=} (\mathbf{i}^* \eta^{\text{bulk}})_{\mu\nu} = (f^* \eta^{\text{bulk}})_{\mu\nu} = \partial_\mu f^A \partial_\nu f^B \eta^{\text{bulk}}_{AB} \quad \text{satisfying} \quad (4.10)$$

$$\gamma_{\mu\nu} v^\mu w^\nu = \eta^{\text{bulk}}(\mathbf{i}_* v, \mathbf{i}_* w) = \eta^{\text{bulk}}_{AB} (f_* v)^A (f_* w)^B \quad \text{for } \forall v, w \in T_p \mathcal{WV}_{\text{sq}}, \quad (4.11)$$

where the two maps f^* and f_* from $f^A = Y^A \circ \mathbf{i} \circ (x^\mu)^{-1}$ are the coordinate representations of the pullback and the pushforward maps \mathbf{i}^* and \mathbf{i}_* (e.g., \mathbf{i}'_p in Eq. (4.5)) in the brane- and bulk-charts x^μ and Y^A (cf. Refs. [3, 12]). Then, $\gamma_{\mu\nu}(x^\rho(p))$ is a tensor defined on the tangent space $T_p \mathcal{WV}_{\text{sq}}$ at a point $p \in \mathcal{WV}_{\text{sq}}$, whereas $\eta^{\text{bulk}}_{AB}(Y^C(p))$ is a tensor defined on $T_p \mathbb{M}^{D_{\text{amb}}}$ at the same point $p = \mathbf{i}(p)$ as an element of $\mathbb{M}^{D_{\text{amb}}}$.

Then, besides the constraint in Eq. (4.4) (equivalently, $\text{rank}(\partial_\mu f^A) = 4$), we assume another constraint on $f^A(x^\mu)$ that the symmetric tensor $\gamma_{\mu\nu} = \partial_\mu f^A \partial_\nu f^B \eta^{\text{bulk}}_{AB}$ in Eq. (4.10) is non-degenerate everywhere on the world volume \mathcal{WV}_{sq} , i.e.,

$$\det(\gamma_{\mu\nu}) \neq 0, \quad (4.12)$$

which means that the pullback $\gamma_{\mu\nu}$ becomes a “metric tensor” on \mathcal{WV}_{sq} (called the “induced metric”). Note that $\det(\gamma_{\mu\nu}) \neq 0$ is a sufficient condition for $\text{rank}(\mathbf{i}) = \text{rank}(\partial_\mu f^A) = 4$ in Eq. (4.4).

Relative to the bulk metric η^{bulk} , the 4D subspace $\mathbf{i}'_p(T_p \mathcal{WV}_{\text{sq}})$ of the bulk tangent space $T_p \mathbb{M}^{D_{\text{amb}}}$ in Eq. (4.5) contains both

- *timelike* bulk-vectors $V_t = \mathbf{i}_* v_t$ (i.e., $\eta^{\text{bulk}}(\mathbf{i}_* v_t, \mathbf{i}_* v_t) < 0$) due to the time evolution in the ambient spacetime $\mathbb{M}^{D_{\text{amb}}}$, and
- *spacelike* bulk-vectors $V_s = \mathbf{i}_* v_s$ (i.e., $\eta^{\text{bulk}}(\mathbf{i}_* v_s, \mathbf{i}_* v_s) > 0$) due to the three-brane nature of the space 3-brane.

Thus, the restriction $\eta^{\text{bulk}}|_{\mathbf{i}'_p(T_p \mathcal{WV}_{\text{sq}})}$ of the bulk-tensor η^{bulk} to the subspace $\mathbf{i}'_p(T_p \mathcal{WV}_{\text{sq}})$ has the (3+1)-dimensional Lorentzian signature $(-, +, +, +)$. This signature $(-, +, +, +)$ is

shared by the induced metric $\gamma_{\mu\nu}$, because $\gamma_{\mu\nu}$ as the pullback of η_{AB}^{bulk} satisfies, for example, $\gamma_{\mu\nu} v_k^\mu v_k^\nu = \eta^{\text{bulk}}(\mathbf{i}_* v_k, \mathbf{i}_* v_k)$ with $k = t, s$ (see Eq. (4.11)).

Therefore, through Eq. (4.11), the induced metric $\gamma_{\mu\nu}$ (i.e., the pullback $f^*(\eta_{AB}^{\text{bulk}})$ of η_{AB}^{bulk} by f^A) becomes a Lorentzian metric having the 4D Lorentzian signature $(-, +, +, +)$. Then, the 4D Lorentzian manifold $(\mathcal{WV}_{\text{sq}}, \gamma_{\mu\nu})$ is interpreted as a (3+1)-dimensional spacetime. This spacetime manifold $(\mathcal{WV}_{\text{sq}}, \gamma_{\mu\nu})$ is an “emergent entity,” because $(\mathcal{WV}_{\text{sq}}, \gamma_{\mu\nu})$ arises through interactions among many space quanta which occupy the ambient spacetime $\mathbb{M}^{D_{\text{amb}}}$.

To sum up, the 4D manifold $(\mathcal{WV}_{\text{sq}}, \gamma_{\mu\nu})$ is the (3+1)-dimensional emergent spacetime which occupies the ambient spacetime $\mathbb{M}^{D_{\text{amb}}}$. Since the spacetime of our universe is (3+1)-dimensional like $(\mathcal{WV}_{\text{sq}}, \gamma_{\mu\nu})$, we assume that the emergent spacetime $(\mathcal{WV}_{\text{sq}}, \gamma_{\mu\nu})$ occupying $\mathbb{M}^{D_{\text{amb}}}$ forms the spacetime of our universe — $(\mathcal{WV}_{\text{sq}}, \gamma_{\mu\nu})$ is the model of our spacetime.

Note that the emergent spacetime $(\mathcal{WV}_{\text{sq}}, \gamma_{\mu\nu})$ is the *exact or true* spacetime of our universe. Thus, when we say that a spacetime and a metric of the universe are observed (or measured), the observed spacetime and the observed metric should be identical to \mathcal{WV}_{sq} and $\gamma_{\mu\nu}$ within the measurement precisions.

Then, since General Relativity has accurately explained the spacetime of our universe, we can think that the spacetime \mathcal{S}_{GR} and the metric $\mathbf{g}_{\mu\nu}$ of General Relativity are at least the good approximations of the world volume \mathcal{WV}_{sq} and the induced metric $\gamma_{\mu\nu}$ (see around Eqs. (7.36) and (7.37)), i.e.,

$$\mathcal{S}_{\text{GR}} \approx \mathcal{WV}_{\text{sq}} , \quad (4.13)$$

$$\mathbf{g}_{\mu\nu} \approx \gamma_{\mu\nu} \quad (= \partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}}) . \quad (4.14)$$

Because Einstein developed General Relativity without considering the space quanta and the ambient spacetime, General Relativity is a *phenomenological* theory of spacetime like the meson theory which Yukawa developed without considering quarks and gluons.

Until now, we have established the kinematics for the space 3-brane: the world volume \mathcal{WV}_{sq} of the space 3-brane in the ambient spacetime $\mathbb{M}^{D_{\text{amb}}}$ is a 4D submanifold of $\mathbb{M}^{D_{\text{amb}}}$, which is described by the brane-to-bulk transformation $Y^A = f^A(x^\mu)$ satisfying

$$(i) \text{ an embedding (i.e., an immersion and an injection) ,} \quad (4.15)$$

$$(ii) \text{ the 4D Lorentzian signature of the induced metric } \gamma_{\mu\nu} = \partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}} . \quad (4.16)$$

In Eq. (4.15), the immersion condition is replaced with $\det(\gamma_{\mu\nu}) \neq 0$ contained in Eq. (4.16) (see below Eq. (4.12)), and the injection condition may be omitted in the case of eccentric behaviors of the space 3-brane (e.g., self-intersections). Note that the function $f^A(x^\mu)$ describing the world volume \mathcal{WV}_{sq} is neither an arbitrary function of x^μ nor a general embedding, but it is a special kind of embedding called the “4D-Lorentzian (4DL) embedding,” which is defined as a function satisfying the conditions in Eqs. (4.15) and (4.16).

Based on the above kinematics for the space 3-brane, we have to consider its dynamics: an effective theory for the space 3-brane can be defined by the “3-brane action”

$$S_{\text{emb}}^{(3\text{br})}[f^A] = \int_{\mathcal{WV}_{\text{sq}}} d^4x \, \widehat{\mathcal{L}}_{\text{emb}}(f^A, \partial_\mu f^A, \dots) , \quad (4.17)$$

where the Lagrangian density $\widehat{\mathcal{L}}_{\text{emb}}$ can contain the UV cutoff Λ_{cont} in Eq. (3.4). In Eq. (4.17), the symbol $\widehat{}$ in $\widehat{\mathcal{L}}_{\text{emb}}$ does not denote the operator nature of $\widehat{\mathcal{L}}_{\text{emb}}$ unlike the same symbol used in, e.g., $\widehat{a}(\vec{k}, \sigma)$ of Sec. 2. Note \mathcal{L}_{emb} without the symbol $\widehat{}$ is called the “Lagrangian” (see below Eq. (4.25)).

Because the full theory of the effective theory $S_{\text{emb}}^{(3\text{br})}[f^A]$ is not known, we use the bottom-up approach to building an effective theory, i.e., writing out the most general set of Lagrangians consistent with the *symmetries* of the theory [11]. Then, the crucial step is to find the symmetries satisfied by the effective action $S_{\text{emb}}^{(3\text{br})}[f^A]$ for the embedding $f^A(x)$.

To find the symmetries of the 3-brane action $S_{\text{emb}}^{(3\text{br})}[f^A]$, we will use a generalization based on the special case of a 0-brane (i.e., point particle) in the 4D Minkowski spacetime \mathbb{M}^4 , as follows: similarly to the space 3-brane in the ambient spacetime $\mathbb{M}^{D_{\text{amb}}}$, the 0-brane in \mathbb{M}^4 produces a 1D world line \mathcal{WL} in \mathbb{M}^4 , which is described by a 1D-Lorentzian embedding $h^\mu(\tau)$ of the world-line parameter τ ($\in \mathbb{R}$).

It is well known that the effective action $S_{\text{emb}}^{(0\text{br})}[h^\mu]$ for the 0-brane has two kinds of symmetries under (i) the 4D Poincaré group $ISO(1, 3)$ with $h'^\mu(\tau) = \Lambda_B^\mu{}_\nu h^\nu(\tau) + c^\mu$, and (ii) the 1D diffeomorphism group $\text{Diff}(1)$ with $h'^\mu(\tau') = h^\mu(\tau)$, where $\tau' = \Phi_{1\text{D}}(\tau)$ for $\Phi_{1\text{D}} \in \text{Diff}(1)$. Note that the $ISO(1, 3)$ symmetry is required by the Special Principle of Relativity (i.e., special covariance) in \mathbb{M}^4 .

Therefore, by using the generalization from the “0-brane in \mathbb{M}^4 ” to a “ p_{br} -brane in \mathbb{M}^D ,” the effective action $S_{\text{emb}}^{(3\text{br})}[f^A]$ for the space 3-brane in $\mathbb{M}^{D_{\text{amb}}}$ (i.e., $p_{\text{br}} = 3$ and $D = D_{\text{amb}}$) has two corresponding symmetries: the first symmetry is the invariance of the 3-brane action $S_{\text{emb}}^{(3\text{br})}[f^A]$ under the bulk Poincaré group $ISO(1, D_{\text{amb}} - 1)$ with

$$f'^A(x^\mu(p)) = \Lambda_B^A f^B(x^\mu(p)) + c^A \quad \text{at a point } p \text{ of } \mathcal{WV}_{\text{sq}} \quad (4.18)$$

$$\text{for } f'^A = Y'^A \circ \mathbf{i} \circ (x^\mu)^{-1} \quad \text{and} \quad f^A = Y^A \circ \mathbf{i} \circ (x^\mu)^{-1}, \quad (4.19)$$

where Λ_B^A and c^A denote each element of the bulk Lorentz group $SO(1, D_{\text{amb}} - 1)$, and each translation in $\mathbb{M}^{D_{\text{amb}}}$, respectively.

Due to Eq. (4.19), the transformation $f^A \rightarrow f'^A$ in Eq. (4.18) uses only the $B \Rightarrow B'$ transformation $Y^A \rightarrow Y'^A$ while keeping the brane-chart x^μ fixed. In other words, the above $ISO(1, D_{\text{amb}} - 1)$ is exactly the same as the set of all coordinate transformations $Y^A \rightarrow Y'^A$ of the ambient manifold $\mathbb{M}^{D_{\text{amb}}}$.

The second one is the invariance of the 3-brane action $S_{\text{emb}}^{(3\text{br})}[f^A]$ under the 4D *local*-reparametrization group $\text{Diff}(4)$ (i.e., the symmetry group of General Relativity) with

$$x'^\mu(p) = \Phi_{4\text{D}}^\mu(x^\nu(p)) \quad \text{and} \quad f'^A(x'^\mu(p)) = f^A(x^\mu(p)) \quad \text{at the same point } p \quad (4.20)$$

$$\text{for } f'^A = Y^A \circ \mathbf{i} \circ (x'^\mu)^{-1} \quad \text{and} \quad f^A = Y^A \circ \mathbf{i} \circ (x^\mu)^{-1}, \quad (4.21)$$

where $\Phi_{4\text{D}} \in \text{Diff}(4)$ corresponds to every general coordinate transformation of General Relativity.

Due to Eq. (4.21), the transformation $f'^A(x') = f^A(x)$ in Eq. (4.20) uses only the $b \Rightarrow b'$ transformation $x^\mu \rightarrow x'^\mu$ while keeping the bulk-chart Y^A fixed. In other words, the above

Diff(4) is exactly the same as the set of all coordinate transformations $x^\mu \rightarrow x'^\mu$ of the sub-manifold \mathcal{WV}_{sq} . The insertion of Eq. (4.21) into Eq. (4.20) produces $Y^A(p) = f'^A(x'(p)) = f^A(x(p))$, which means the invariance of $Y^A(p)$ under $x(p) \rightarrow x'(p)$. This transformation law $f'^A(x') = f^A(x)$ under $x \rightarrow x'$ implies that each of $f^{A=0,\dots,D_{\text{amb}}-1}$ is a scalar field under Diff(4). Note that the Nambu-Goto action for a bosonic string is the $p_{\text{br}} = 1$ case in the above generalization, having the similar kinds of symmetries [5, 7].

First, we deal with the Diff(4) invariance of the 3-brane action $S_{\text{emb}}^{(3\text{br})}[f^A]$ more closely: since the world volume \mathcal{WV}_{sq} exists in the ambient spacetime $\mathbb{M}^{D_{\text{amb}}}$ irrespective of the $b \Rightarrow b'$ transformation $x \rightarrow x' = \Phi_{4D}(x)$ in Eq. (4.20), the pair $f'^A(x')$ and $f^A(x)$ should be simultaneously the solutions for the equation of motion. Thus, if the unprimed map $f^A(x)$ is an extremum point of the unprimed action $S_{\text{emb}}^{(3\text{br})}[f^A]$ (i.e., $f^A(x)$ obeys Hamilton's principle $\delta S_{\text{emb}}^{(3\text{br})}[f^A] = 0$), then the primed map $f'^A(x')$ is an extremum point of the primed action $S_{\text{emb}}^{(3\text{br})'}[f'^A]$ in the primed system $(x'^\rho, f'^A, \hat{\mathcal{L}}'_{\text{emb}})$

$$S_{\text{emb}}^{(3\text{br})'}[f'^A] = \int_{\mathcal{WV}_{\text{sq}}} d^4x' \hat{\mathcal{L}}'_{\text{emb}}(f'^A, \partial'_\rho f'^A, \dots), \quad (4.22)$$

and vice versa.

The situation that both of $f'^A(x')$ and $f^A(x)$ are the solutions can be realized by the sameness in the values of the two actions (called the “value invariance of the action”)

$$S_{\text{emb}}^{(3\text{br})'}[f'^A] = S_{\text{emb}}^{(3\text{br})}[f^A], \quad (4.23)$$

which leads to

$$\hat{\mathcal{L}}'_{\text{emb}}(f'^A, \partial'_\rho f'^A, \dots) = \det(\partial x^\mu / \partial x'^\rho) \times \hat{\mathcal{L}}_{\text{emb}}(f^A, \partial_\mu f^A, \dots). \quad (4.24)$$

Due to $f'^A(x') = f^A(x)$ in Eq. (4.20), the primed metric $\gamma'_{\rho\sigma} \stackrel{\text{def}}{=} \partial'_\rho f'^A \partial'_\sigma f'^B \eta_{AB}^{\text{bulk}}$ follows the usual transformation law $\gamma'_{\rho\sigma} = \frac{\partial x^\mu}{\partial x'^\rho} \frac{\partial x^\nu}{\partial x'^\sigma} \gamma_{\mu\nu}$ for a (0, 2)-type tensor, which together with $\det(\gamma_{\mu\nu}) \neq 0$ in Eq. (4.12) implies

$$\det(\partial x^\mu / \partial x'^\rho) = \sqrt{|\det(\gamma'_{\rho\sigma})|} / \sqrt{|\det(\gamma_{\mu\nu})|}. \quad (4.25)$$

Then, the Lagrangian \mathcal{L}_{emb} defined as $\mathcal{L}_{\text{emb}} \stackrel{\text{def}}{=} \hat{\mathcal{L}}_{\text{emb}} / \sqrt{|\det(\gamma_{\mu\nu})|}$ is a scalar under Diff(4) due to $\mathcal{L}'_{\text{emb}}(f'^A, \partial'_\rho f'^A, \dots) = \mathcal{L}_{\text{emb}}(f^A, \partial_\mu f^A, \dots)$ unlike the scalar density $\hat{\mathcal{L}}_{\text{emb}}$ of weight -1 . Note that $\sqrt{|\det(\gamma_{\mu\nu})|}$ is a function of $\partial_\mu f^A$ due to the definition $\gamma_{\mu\nu} = \partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}}$ in Eq. (4.10).

In addition, when the “form invariance of the Lagrangian density”

$$\hat{\mathcal{L}}'_{\text{emb}}(f'^A, \partial'_\rho f'^A, \dots) = \hat{\mathcal{L}}_{\text{emb}}(f'^A, \partial'_\rho f'^A, \dots) \quad (4.26)$$

(thus $\mathcal{L}'_{\text{emb}}(f'^A, \partial'_\rho f'^A, \dots) = \mathcal{L}_{\text{emb}}(f'^A, \partial'_\rho f'^A, \dots)$) is fulfilled, the Euler-Lagrange equation for the primed map $f'^A(x')$ has the same form as that for the unprimed map $f^A(x)$.

Similarly, the $ISO(1, D_{\text{amb}} - 1)$ invariance of $S_{\text{emb}}^{(3\text{br})}[f^A]$ consists of two parts, (a) the value invariance of the action, and (b) the form invariance of the Lagrangian density. Due to these value and form invariances, the invariance under a translation $f^A \rightarrow f^A + c^A$ from $ISO(1, D_{\text{amb}} - 1)$ means that the Lagrangian density $\hat{\mathcal{L}}_{\text{emb}}$ (thus \mathcal{L}_{emb}) does not contain any derivative-free terms of f^A (e.g., $f^A f^B \eta_{AB}^{\text{bulk}}$), implying $\hat{\mathcal{L}}_{\text{emb}} = \hat{\mathcal{L}}_{\text{emb}}(\partial_\mu f^A, \dots)$.

From now on, the effective action $S_{\text{emb}}^{(3\text{br})}[f^A]$ in Eq. (4.17) is expressed as the form using the Diff(4)-invariant volume element $d^4x \sqrt{|\det(\gamma_{\mu\nu})|}$

$$S_{\text{emb}}^{(3\text{br})}[f^A] = \int_{\mathcal{WV}_{\text{sq}}} d^4x \sqrt{|\det(\gamma_{\mu\nu})|} \mathcal{L}_{\text{emb}}(\partial_\mu f^A, \dots), \quad (4.27)$$

where the Lagrangian \mathcal{L}_{emb} can contain the UV cutoff Λ_{cont} in Eq. (3.4). The Lagrangian \mathcal{L}_{emb} of the 3-brane action $S_{\text{emb}}^{(3\text{br})}[f^A]$ can have the form of

$$\mathcal{L}_{\text{emb}}(\partial_\mu f^A, \dots) = -\mathcal{T}_{3\text{br}} + \mathcal{L}_{\text{emb}}^{(\text{der})}(\partial_\mu f^A, \dots), \quad (4.28)$$

where $\mathcal{T}_{3\text{br}}$ is the “energy density” or “three-brane tension” of the space 3-brane, which corresponds to the Nambu-Goto action for a three-brane [5, 7].

The “derivative Lagrangian” $\mathcal{L}_{\text{emb}}^{(\text{der})}(\partial_\mu f^A, \dots)$ in Eq. (4.28) does not have any constant term. This derivative Lagrangian $\mathcal{L}_{\text{emb}}^{(\text{der})}$ can be originated from (i) internal interactions between space quanta (e.g., elastic forces), and/or (ii) external interactions of space quanta with other kind(s) of particles. This Lagrangian $\mathcal{L}_{\text{emb}}^{(\text{der})}$ can contain the Einstein-Hilbert term $d_2 \Lambda_{\text{cont}}^2 R$, where R is the Ricci scalar built from the induced metric $\gamma_{\mu\nu}$ (d_2 : constant).

The “embedding scalars” f^A of the 3-brane action $S_{\text{emb}}^{(3\text{br})}[f^A]$ are *different* in two ways from ordinary scalars (e.g., pions $\pi^{\pm,0}$) in General Relativity, as follows:

First, unlike the ordinary scalars, the embedding scalars $f^A(x)$ appear in the metric tensor $\gamma_{\mu\nu}$ of the world volume \mathcal{WV}_{sq} through the definition $\gamma_{\mu\nu} = \partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}}$. As a result, the embedding scalars f^A also appear in the quantities depending on $\gamma_{\mu\nu}$, for example, (i) $\sqrt{|\det(\gamma_{\mu\nu})|}$ in the action $S_{\text{emb}}^{(3\text{br})}[f^A]$, (ii) the Christoffel symbols $\Gamma_{\mu\nu}^\rho$ for the covariant derivative ∇_μ , and (iii) the inverse metric $\gamma^{\rho\sigma}$ for contraction.

Second, unlike the ordinary scalars, the solution $f_{\text{sol}}^A(x^\mu)$ of the equation $\delta S_{\text{emb}}^{(3\text{br})}[f^A] = 0$ is not an arbitrary function, but a 4DL embedding. This 4DL embedding $f_{\text{sol}}^A(x^\mu)$ makes the induced metric $\gamma_{\mu\nu}$ a 4D metric of the signature $(-, +, +, +)$. Then, the non-zero value of the composite field $\gamma_{\mu\nu} = \partial_\mu f_{\text{sol}}^A \partial_\nu f_{\text{sol}}^B \eta_{AB}^{\text{bulk}}$ may be interpreted as the “condensation” for the covariant four-vectors $\partial_\mu f^A$.

Now, we want to find the 4D-Lorentzian embedding $f^A(x)$ which makes the world volume \mathcal{WV}_{sq} *globally flat*, that is, the induced metric

$$\gamma_{\mu\nu}(x^\rho(p)) = \eta_{\mu\nu} \quad \text{at every point } p \text{ of } \mathcal{WV}_{\text{sq}}. \quad (4.29)$$

Due to the definition of $\gamma_{\mu\nu}$, the equality $\gamma_{\mu\nu} = \eta_{\mu\nu}$ in Eq. (4.29) can be represented as the partial differential equation (PDE) for the 4DL embedding f^A

$$\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}} = \eta_{\mu\nu}. \quad (4.30)$$

This PDE for the embedding f^A can be solved, when its derivatives $\partial_\mu f^A$ satisfy

$$\partial_\mu f^A = \Lambda_{B^\mu}^A \quad \text{everywhere on } \mathcal{WV}_{\text{sq}}, \quad (4.31)$$

where $\Lambda_B^A \in SO(1, D_{\text{amb}} - 1)$, $B^0 = 0$ (i.e., the bulk time), and three different $B^{i=1,2,3} \in \{1, \dots, D_{\text{amb}} - 1\}$. A simple example is $\Lambda_{B^\mu}^A = \delta_\mu^A$, where δ_μ^A comes from the bulk Kronecker delta. For $\partial_\mu f^A = \Lambda_{B^\mu}^A$ in Eq. (4.31), the induced metric is expressed as

$$\gamma_{\mu\nu} = \eta_{B^\mu B^\nu}^{\text{bulk}}, \quad (4.32)$$

which corresponds to the 4D Minkowski spacetime \mathbb{M}^4 .

Because $\partial_\nu \Lambda_{B^\mu}^A = 0$ everywhere on \mathcal{WV}_{sq} , the integration of Eq. (4.31) leads to a linear function of x^μ

$$f_{\text{lin}}^A(x) = \Lambda_{B^\mu}^A x^\mu + D^A, \quad (4.33)$$

where D^A are independent of x^μ .

Then, our remaining task is to check whether this linear embedding $f_{\text{lin}}^A(x)$ is a solution of the Euler-Lagrange (E-L) equation, as follows: Hamilton's principle using the Lagrangian $\mathcal{L}_{\text{emb}} = -\mathcal{T}_{3\text{br}} + \mathcal{L}_{\text{emb}}^{(\text{der})}$ in Eq. (4.28)

$$\frac{\delta S_{\text{emb}}^{(3\text{br})}}{\delta f^A} [f^A] = 0 \quad (4.34)$$

produces the equation of motion for the space 3-brane (i.e., the E-L equation)

$$\left\{ \partial_\mu \left[\mathcal{T}_{3\text{br}} \sqrt{|\det(\gamma_{\rho\sigma})|} \gamma^{\mu\nu} \partial_\nu f^B \eta_{AB}^{\text{bulk}} \right] \right\} + \dots = 0 \quad \text{with } \gamma_{\rho\sigma} = \partial_\rho f^A \partial_\sigma f^B \eta_{AB}^{\text{bulk}}, \quad (4.35)$$

where the ellipsis \dots denotes the contribution of the derivative Lagrangian $\mathcal{L}_{\text{emb}}^{(\text{der})}$.

If the energy density $\mathcal{T}_{3\text{br}}$ of the space 3-brane is independent of x^μ , then each term of Eq. (4.35) can contain only the second or higher derivatives of f^A (e.g., $\partial_\mu \partial_\nu f^A$). As a result, each term of Eq. (4.35) vanishes for any linear function of x^μ , for example, the linear embedding $f_{\text{lin}}^A(x) = \Lambda_{B^\mu}^A x^\mu + D^A$ in Eq. (4.33).

Therefore, for the x^μ -independent energy density $\mathcal{T}_{3\text{br}}$, the linear embedding $f_{\text{lin}}^A(x)$ is the solution of the E-L equation in Eq. (4.35), implying the world volume \mathcal{WV}_{sq} is the 4D Minkowski spacetime \mathbb{M}^4 due to the flat metric $\gamma_{\mu\nu} = \eta_{B^\mu B^\nu}^{\text{bulk}}$ induced by $f_{\text{lin}}^A(x)$.

Since the linear embedding $f_{\text{lin}}^A(x)$ satisfies the E-L equation *irrespective of the value* of the x^μ -independent $\mathcal{T}_{3\text{br}}$, the flat metric $\gamma_{\mu\nu} = \eta_{\mu\nu}$ of the world volume \mathcal{WV}_{sq} exists for any value of the uniform energy density $\mathcal{T}_{3\text{br}}$. This interesting feature distinguishes the 3-brane action $S_{\text{emb}}^{(3\text{br})}[f^A]$ from General Relativity.

5 The Aim-At-Target (AAT) Method for Studying the World Volume of the Space 3-Brane

In Sec. 4, since the space 3-brane occupies the ambient spacetime $\mathbb{M}^{D_{\text{amb}}}$, the effective theory $S_{\text{emb}}^{(3\text{br})}[f^A]$ of the space 3-brane was built for the ambient spacetime $\mathbb{M}^{D_{\text{amb}}}$ through the field $f^A(x^\mu) \in \mathbb{M}^{D_{\text{amb}}}$. The world volume $(\mathcal{WV}_{\text{sq}}, \gamma_{\mu\nu})$ of the space 3-brane is described by the solution f_{sol}^A for the equation of motion $\delta S_{\text{emb}}^{(3\text{br})}/\delta f^A = 0$. Since this world volume $(\mathcal{WV}_{\text{sq}}, \gamma_{\mu\nu})$ is the exact or true spacetime of our universe, it is important to know the solution f_{sol}^A describing our spacetime $(\mathcal{WV}_{\text{sq}}, \gamma_{\mu\nu})$.

In this section, we want to show a methodology for studying the world volume $(\mathcal{WV}_{\text{sq}}, \gamma_{\mu\nu})$ by using a *metric action* $S_{\text{met}}[g_{\mu\nu}]$ (see Table 1). The key point of this methodology is that the solution f_{sol}^A of $\delta S_{\text{emb}}^{(3\text{br})}/\delta f^A = 0$ can be found by solving the different equation $\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}} = g_{\mu\nu}$ when the new metric $g_{\mu\nu}$ satisfies $g_{\mu\nu} = \gamma_{\mu\nu}$ ($= \partial_\mu f_{\text{sol}}^A \partial_\nu f_{\text{sol}}^B \eta_{AB}^{\text{bulk}}$). An example of the methodology is the flat-metric case $\gamma_{\mu\nu} = \eta_{\mu\nu}$, whose treatment is shown below Eq. (4.29). The details of our methodology are shown, as follows:

To study the world volume $(\mathcal{WV}_{\text{sq}}, \gamma_{\mu\nu})$ of the space 3-brane, we consider the solution set Σ_{target} for the equation of motion $\frac{\delta S_{\text{emb}}^{(3\text{br})}}{\delta f^A}[f^A] = 0$ in Eq. (4.34)

$$\Sigma_{\text{target}} \stackrel{\text{def}}{=} \{ f_{\text{sol}}^A : 4\text{DL embedding} \mid (\delta S_{\text{emb}}^{(3\text{br})}/\delta f^A)[f_{\text{sol}}^A] = 0 \} \subset \mathcal{F}_{\text{space}}, \quad (5.1)$$

where $\mathcal{F}_{\text{space}} = \{\phi^A\}$ is the function space, and the element f_{sol}^A is called the “4D-Lorentzian (4DL) solution.” For the given **original action** $S_{\text{emb}}^{(3\text{br})}[f^A]$, the solution set $\Sigma_{\text{target}} = \{f_{\text{sol}}^A\}$ is a *fixed* set in the function space $\mathcal{F}_{\text{space}}$. Note that $\Sigma_{\text{target}} \ni f_{\text{lin}}^A$ for a uniform energy density $\mathcal{T}_{3\text{br}}$ (see around Eq. (4.34)).

By the way, since our methodology to study the solution set $\Sigma_{\text{target}} = \{f_{\text{sol}}^A\}$ is similar to “a bullet fired at a fixed target in space” (see the discussions around Eq. (5.13)), the solution set $\Sigma_{\text{target}} = \{f_{\text{sol}}^A\}$ is called the **target** (or target set) in the **space** $\mathcal{F}_{\text{space}} = \{\phi^A\}$. For an easy understanding, we show the tenors and the vehicles in the bullet-target (B-T) metaphor

$$\langle \Sigma_{\text{bullet}}, \Sigma_{\text{target}}, \mathcal{F}_{\text{space}} \rangle \iff \langle \langle \text{bullet}, \text{target}, \text{space} \rangle \rangle, \quad (5.2)$$

where the “bullet” Σ_{bullet} will be defined in Eq. (5.9). Moreover, there are discussions about the “rifle” (above Eq. (5.10)) and the “aim-of-the-rifle” (above Eq. (5.13)).

Through the definition $\gamma_{\mu\nu} = \partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}}$ in Eq. (4.10), the target $\Sigma_{\text{target}} = \{f_{\text{sol}}^A\}$ in Eq. (5.1) produces the set Σ_{ind} of the induced metrics $\gamma_{\mu\nu}$ with the signature $(-, +, +, +)$

$$\Sigma_{\text{ind}} \stackrel{\text{def}}{=} \{ \gamma_{\mu\nu} \mid \gamma_{\mu\nu} = \partial_\mu f_{\text{sol}}^A \partial_\nu f_{\text{sol}}^B \eta_{AB}^{\text{bulk}} \text{ for every } f_{\text{sol}}^A \in \Sigma_{\text{target}} \}. \quad (5.3)$$

Note that $\Sigma_{\text{ind}} \ni \eta_{\mu\nu}$ for a uniform energy density $\mathcal{T}_{3\text{br}}$ (see around Eq. (4.32)).

Conversely, this “induced-metric set” $\Sigma_{\text{ind}} = \{\gamma_{\mu\nu}\}$ can produce the target set $\Sigma_{\text{target}} = \{f_{\text{sol}}^A\}$, because (i) the former set $\Sigma_{\text{ind}} = \{\gamma_{\mu\nu}\}$ produces the solution set $\Sigma_{\text{PDE}}^{(\text{sol})}$ of the partial differential equation (PDE) for the embedding f^A

$$\Sigma_{\text{PDE}}^{(\text{sol})} \stackrel{\text{def}}{=} \{ f^A \mid \partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}} = \gamma_{\mu\nu} \text{ for every } \gamma_{\mu\nu} \in \Sigma_{\text{ind}} \} , \quad (5.4)$$

and (ii) this “PDE solution set” $\Sigma_{\text{PDE}}^{(\text{sol})}$ contains the target set $\Sigma_{\text{target}} = \{f_{\text{sol}}^A\}$, i.e.,

$$\Sigma_{\text{PDE}}^{(\text{sol})} \supset \Sigma_{\text{target}} . \quad (5.5)$$

The result $\Sigma_{\text{PDE}}^{(\text{sol})} \supset \Sigma_{\text{target}}$ in Eq. (5.5) suggests a hint about how to know the target set $\Sigma_{\text{target}} (\subset \mathcal{F}_{\text{space}})$, implying the importance of studying the induced-metric set $\Sigma_{\text{ind}} = \{\gamma_{\mu\nu}\}$. To study this set $\Sigma_{\text{ind}} = \{\gamma_{\mu\nu}\}$, since its element $\gamma_{\mu\nu}$ is a 4D Lorentzian metric, we consider the theory $S_{\text{met}}[g_{\mu\nu}]$ of a 4D Lorentzian metric $g_{\mu\nu}$ (rather than $\gamma_{\mu\nu}$)

$$S_{\text{met}}[g_{\mu\nu}] = \int_{\mathcal{S}_{\text{met}}^{4\text{D}}} d^4x \sqrt{|\det(g_{\mu\nu})|} \mathcal{L}_{\text{met}}(\Lambda_{\text{met}}; g_{\mu\nu}) , \quad (5.6)$$

where $(\mathcal{S}_{\text{met}}^{4\text{D}}, g_{\mu\nu})$ is a 4D spacetime manifold, $\mathcal{L}_{\text{met}}(\Lambda_{\text{met}}; g_{\mu\nu})$ is the Lagrangian containing the derivatives of $g_{\mu\nu}$, and Λ_{met} is the UV cutoff of the metric action $S_{\text{met}}[g_{\mu\nu}]$ (see Table 1 for the role of $S_{\text{met}}[g_{\mu\nu}]$).

Since the metric action $S_{\text{met}}[g_{\mu\nu}]$ in Eq. (5.6) neglects the *microscopic* behaviors of individual space quanta (i.e., the underlying discreteness of the space 3-brane) like the original action $S_{\text{emb}}^{(3\text{br})}[f^A]$, we can assume the UV cutoff Λ_{met} of the metric action $S_{\text{met}}[g_{\mu\nu}]$ satisfies

$$\Lambda_{\text{met}} \lesssim O(\Lambda_{\text{cont}}) , \quad (5.7)$$

where Λ_{cont} is the UV cutoff of the original action $S_{\text{emb}}^{(3\text{br})}[f^A]$ (see Eqs. (3.4) and (4.17)). The spacetime manifold $(\mathcal{S}_{\text{met}}^{4\text{D}}, g_{\mu\nu})$ for the metric action $S_{\text{met}}[g_{\mu\nu}]$ can be a good approximation of the exact or true spacetime manifold $(\mathcal{WV}_{\text{sq}}, \gamma_{\mu\nu})$, which is an emergent object arising from many space quanta in $\mathbb{M}^{D_{\text{amb}}}$ (see Eqs. (6.9), (6.10) and (6.11)).

Then, we can approach the induced-metric set $\Sigma_{\text{ind}} = \{\gamma_{\mu\nu}\}$ by using the solution set $\Sigma_{\text{met}}^{(\text{sol})}$ of the equation $\frac{\delta S_{\text{met}}}{\delta g_{\mu\nu}}[g_{\mu\nu}] = 0$

$$\Sigma_{\text{met}}^{(\text{sol})} \stackrel{\text{def}}{=} \{ g_{\mu\nu}^{\text{sol}} \mid (\delta S_{\text{met}} / \delta g_{\mu\nu})[g_{\mu\nu}^{\text{sol}}] = 0 \} , \quad (5.8)$$

which is called the **cartridge** (see above Eq. (5.10)). The solution $g_{\mu\nu}^{\text{sol}}$ in Eq. (5.8) is called the “solution metric.”

In order to study the target $\Sigma_{\text{target}} = \{f_{\text{sol}}^A\}$ in the space $\mathcal{F}_{\text{space}}$, we define the **bullet** Σ_{bullet} (corresponding to the PDE solution set $\Sigma_{\text{PDE}}^{(\text{sol})}$ in Eq. (5.4))

$$\Sigma_{\text{bullet}} \stackrel{\text{def}}{=} \{ f^A \mid \partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}} = g_{\mu\nu}^{\text{sol}} \text{ for every } g_{\mu\nu}^{\text{sol}} \in \Sigma_{\text{met}}^{(\text{sol})} \} \subset \mathcal{F}_{\text{space}} \quad (5.9)$$

by using the cartridge $\Sigma_{\text{met}}^{(\text{sol})} = \{g_{\mu\nu}^{\text{sol}}\}$. Since the bullet set $\Sigma_{\text{bullet}} = \{f_{\text{bul}}^A\}$ can overlap the target set $\Sigma_{\text{target}} = \{f_{\text{sol}}^A\}$ in the function space $\mathcal{F}_{\text{space}} = \{\phi^A\}$ like the “bullet” of the B-T metaphor “a bullet fired at a fixed target in space” (see Eq. (5.2)), the set $\Sigma_{\text{bullet}} = \{f_{\text{bul}}^A\}$ is called the bullet.

In Eq. (5.9), the PDE $\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}} = g_{\mu\nu}^{\text{sol}}$ defines a transformation $\Sigma_{\text{met}}^{(\text{sol})} \rightarrow \Sigma_{\text{bullet}} = \Psi_{\text{PDE}}(\Sigma_{\text{met}}^{(\text{sol})})$ like the rifle of the above B-T metaphor, which transforms its cartridge into the metallic bullet. Thus, the “solution-metric set” $\Sigma_{\text{met}}^{(\text{sol})} = \{g_{\mu\nu}^{\text{sol}}\}$ and the PDE $\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}} = g_{\mu\nu}^{\text{sol}}$ are called the cartridge (see below Eq. (5.8)), and the **rifle** firing the bullet $\Sigma_{\text{bullet}} = \{f_{\text{bul}}^A\}$, respectively. This “rifle PDE,” and the PDE for $\Sigma_{\text{PDE}}^{(\text{sol})}$ in Eq. (5.4) have the same form

$$\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}} = q_{\mu\nu} \quad (q_{\mu\nu} = g_{\mu\nu}^{\text{sol}}, \gamma_{\mu\nu}), \quad (5.10)$$

which is invariant under the bulk Poincaré group $ISO(1, D_{\text{amb}} - 1)$ with $f'^A = \Lambda_B^A f^B + c^A$ (see Sec. 4).

Thus, if the “metric intersection (MI)” $\Sigma_{\text{MI}} \stackrel{\text{def}}{=} \Sigma_{\text{met}}^{(\text{sol})} \cap \Sigma_{\text{ind}} = \{g_{\mu\nu}^{\text{MI}}\}$ contains an element

$$g_{\mu\nu}^{\text{MI}\otimes} = g_{\mu\nu}^{\text{sol}\otimes} = \gamma_{\mu\nu}^{\otimes} \quad (= \partial_\mu f_{\text{sol}\otimes}^A \partial_\nu f_{\text{sol}\otimes}^B \eta_{AB}^{\text{bulk}} \quad \text{with } f_{\text{sol}\otimes}^A \in \Sigma_{\text{target}} = \{f_{\text{sol}}^A\}) , \quad (5.11)$$

then the bullet $\Sigma_{\text{bullet}} = \{f_{\text{bul}}^A\}$ shares the 4DL solution $f_{\text{sol}\otimes}^A$ with the target $\Sigma_{\text{target}} = \{f_{\text{sol}}^A\}$. Since the converse of this proposition is true, we have the equivalence that

$$\Sigma_{\text{MI}} = \Sigma_{\text{met}}^{(\text{sol})} \cap \Sigma_{\text{ind}} \neq \emptyset \quad \text{if and only if} \quad \Sigma_{\text{B}\cap\text{T}} \stackrel{\text{def}}{=} \Sigma_{\text{bullet}} \cap \Sigma_{\text{target}} \neq \emptyset , \quad (5.12)$$

where $\Sigma_{\text{B}\cap\text{T}}$ is called the “bullet-target (B-T) overlap.”

In Eq. (5.12), the B-T overlap $\Sigma_{\text{B}\cap\text{T}} = \Sigma_{\text{bullet}} \cap \Sigma_{\text{target}}$ describes the manner in which the target $\Sigma_{\text{target}} = \{f_{\text{sol}}^A\}$ of the original action $S_{\text{emb}}^{(3\text{br})}[f^A]$ is overlapped by the bullet $\Sigma_{\text{bullet}} = \{f_{\text{bul}}^A\}$ of the metric action $S_{\text{met}}[g_{\mu\nu}]$. Since the original action $S_{\text{emb}}^{(3\text{br})}[f^A]$ is given to us (i.e., not changed arbitrarily by us), the target Σ_{target} in the B-T overlap $\Sigma_{\text{B}\cap\text{T}} = \Sigma_{\text{bullet}} \cap \Sigma_{\text{target}}$ is treated as a *fixed set* in the function space $\mathcal{F}_{\text{space}} = \{\phi^A\}$. Then, the B-T overlap $\Sigma_{\text{B}\cap\text{T}}$ represents the *maximum knowledge about the fixed target* Σ_{target} which we can obtain by using the bullet Σ_{bullet} of the chosen action $S_{\text{met}}[g_{\mu\nu}]$.

For example, the maximum B-T overlap $\Sigma_{\text{B}\cap\text{T}}^{(\text{max})} = \Sigma_{\text{target}}$ (i.e., $\Sigma_{\text{bullet}}^{(\text{max})} \supset \Sigma_{\text{target}}$) means that we can know the whole of the target Σ_{target} by using the maximum bullet $\Sigma_{\text{bullet}}^{(\text{max})}$. However, the minimum overlap $\Sigma_{\text{B}\cap\text{T}}^{(\text{min})} = \emptyset$ (i.e., $\Sigma_{\text{bullet}}^{(\text{min})} \cap \Sigma_{\text{target}} = \emptyset$) means that we cannot know the target Σ_{target} through the minimum bullet $\Sigma_{\text{bullet}}^{(\text{min})}$.

Fortunately, unlike the target Σ_{target} , the bullet Σ_{bullet} changes *depending on* which metric action $S_{\text{met}}[g_{\mu\nu}]$ we choose (see Eqs. (5.8) and (5.9)). Thus, this metric action $S_{\text{met}}[g_{\mu\nu}]$ corresponds to “the aim of the rifle at the target” of the above B-T metaphor “a bullet fired at a fixed target in space.” Then, the metric action $S_{\text{met}}[g_{\mu\nu}]$ is called the **aim-of-the-rifle**.

Through the dependence of Σ_{bullet} on $S_{\text{met}}[g_{\mu\nu}]$, the B-T overlap $\Sigma_{\text{B}\cap\text{T}} = \Sigma_{\text{bullet}} \cap \Sigma_{\text{target}}$ in Eq. (5.12) depends on the metric action $S_{\text{met}}[g_{\mu\nu}]$. In other words, the metric action $S_{\text{met}}[g_{\mu\nu}]$

Table 1: The Outline of the Aim-At-Target (AAT) Method for $\Sigma_{B \cap T} \neq \emptyset$

ACTION	NAME	ROLE
$S_{\text{emb}}^{(3\text{br})}[f^A]$	Original action	The “true theory” of the space 3-brane within $\mathbb{M}^{D_{\text{amb}}}$
$S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}]$	Overlapping action	$\left\{ \begin{array}{l} \text{(a) ME : a “mere tool” for knowing } \Sigma_{\text{target}} \text{ of } S_{\text{emb}}^{(3\text{br})} \\ \text{(b) PE : producing a “constitutive equation”} \end{array} \right.$

$$\left[\begin{array}{l} \textit{Caution} : \text{ the motion of the space 3-brane in } \mathbb{M}^{D_{\text{amb}}} \text{ is described by either} \\ \text{(a')} \text{ the target } \Sigma_{\text{target}} \text{ or (b')} \text{ the B-T overlap } \Sigma_{B \cap T}^{(\text{ovlp}) \cdot \text{PE}}, \text{ depending on} \\ \text{the role of the overlapping action } S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}] \text{ (see the text).} \end{array} \right]$$

produces the bullet Σ_{bullet} , and this bullet Σ_{bullet} produces the B-T overlap $\Sigma_{B \cap T}$, i.e.,

$$S_{\text{met}}[g_{\mu\nu}] \longrightarrow \Sigma_{\text{bullet}} \longrightarrow \Sigma_{B \cap T} (= \Sigma_{\text{bullet}} \cap \Sigma_{\text{target}}). \quad (5.13)$$

Since this sequence in Eq. (5.13) implies that the metric action $S_{\text{met}}[g_{\mu\nu}]$ determines the B-T overlap $\Sigma_{B \cap T}$, various metric actions $S_{\text{met}}[g_{\mu\nu}]$ are classified by their B-T overlap $\Sigma_{B \cap T}$ into two types, (i) the “overlapping” metric action $S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}]$ satisfying $\Sigma_{B \cap T} \neq \emptyset$, and (ii) the “non-overlapping” metric action $S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}]$ satisfying $\Sigma_{B \cap T} = \emptyset$.

Because a non-overlapping action $S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}]$ has its empty B-T overlap $\Sigma_{B \cap T}^{(\text{ovlp})} = \emptyset (= \Sigma_{B \cap T}^{(\text{min})})$, the solution set $\Sigma_{\text{target}} = \{f_{\text{sol}}^A\}$ of the original action $S_{\text{emb}}^{(3\text{br})}[f^A]$ cannot be known by using the non-overlapping bullet $\Sigma_{\text{bullet}}^{(\text{ovlp})}$ (see above). Thus, this *undesirable* action $S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}]$ should be modified.

Then, through the dependence of Σ_{bullet} on $S_{\text{met}}[g_{\mu\nu}]$, we can find a suitable overlapping action $S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}]$ by making $\Sigma_{B \cap T} \neq \emptyset$, namely, by changing (i) the form of the metric action $S_{\text{met}}[g_{\mu\nu}]$ and (ii) the values of its parameters. As a result, we can (partially) know the target $\Sigma_{\text{target}} = \{f_{\text{sol}}^A\}$ by using the bullet $\Sigma_{\text{bullet}}^{(\text{ovlp})}$ of the overlapping action $S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}]$. This methodology for knowing the target Σ_{target} by trying the aim-of-the-rifle $S_{\text{met}}[g_{\mu\nu}]$ is called the “Aim-At-Target (AAT) method.”

For a better understanding of this AAT method, we summarize it for a non-empty B-T overlap $\Sigma_{B \cap T} \neq \emptyset$ (see Table 1), as follows: first, the AAT method uses the two actions $S_{\text{emb}}^{(3\text{br})}[f^A]$ and $S_{\text{met}}[g_{\mu\nu}]$. Second, since the original action $S_{\text{emb}}^{(3\text{br})}[f^A]$ shows the truth (e.g., the equation of motion in $\mathbb{M}^{D_{\text{amb}}}$) about the space 3-brane occupying $\mathbb{M}^{D_{\text{amb}}}$, the 3-brane action $S_{\text{emb}}^{(3\text{br})}[f^A]$ is the “true theory” of the space 3-brane in $\mathbb{M}^{D_{\text{amb}}}$.

Third, when the B-T overlap $\Sigma_{B \cap T} = \Sigma_{\text{bullet}} \cap \Sigma_{\text{target}}$ is not empty (i.e., $\Sigma_{B \cap T} \neq \emptyset$), the metric action $S_{\text{met}}[g_{\mu\nu}]$ can be desirable. Depending on the “mode of existence” (mathematical/physical), the overlapping metric action $S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}]$ has two different implications:

- (a) An overlapping action $S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}]$ has only the “mathematical existence (ME)”

unlike the original action $S_{\text{emb}}^{(3\text{br})}[f^A]$: this overlapping action is called the “ME action” $S_{\text{met}}^{(\text{ME})}[g_{\mu\nu}]$. Due to its mathematical existence, this ME action $S_{\text{met}}^{(\text{ME})}[g_{\mu\nu}]$ cannot affect the occurrence of any element f_{sol}^A of the target $\Sigma_{\text{target}} = \{f_{\text{sol}}^A\}$ through the B-T overlap $\Sigma_{\text{B}\cap\text{T}}^{(\text{ovlp})\cdot\text{ME}}$. In other words, irrespective of the B-T overlap $\Sigma_{\text{B}\cap\text{T}}^{(\text{ovlp})\cdot\text{ME}}$, every element f_{sol}^A of the target Σ_{target} still can occur in the ambient spacetime $\mathbb{M}^{D_{\text{amb}}}$ as a motion of the space 3-brane. Therefore, the ME action $S_{\text{met}}^{(\text{ME})}[g_{\mu\nu}]$ making the overlap $\Sigma_{\text{B}\cap\text{T}}^{(\text{ovlp})\cdot\text{ME}}$ is a “mere tool” for knowing the target Σ_{target} of the true theory $S_{\text{emb}}^{(3\text{br})}[f^A]$.

- (b) An overlapping action $S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}]$ has the “physical existence (PE)” like the original action $S_{\text{emb}}^{(3\text{br})}[f^A]$: this overlapping action is called the “PE action” $S_{\text{met}}^{(\text{PE})}[g_{\mu\nu}]$. Due to its physical existence, this PE action $S_{\text{met}}^{(\text{PE})}[g_{\mu\nu}]$ allows (forbids) the occurrence of an element f_{sol}^A of the target Σ_{target} , when this 4DL solution f_{sol}^A does (not) belong to the B-T bullet $\Sigma_{\text{B}\cap\text{T}}^{(\text{ovlp})\cdot\text{PE}}$. In other words, only the element f_{ove}^A of the B-T overlap $\Sigma_{\text{B}\cap\text{T}}^{(\text{ovlp})\cdot\text{PE}} = \{f_{\text{ove}}^A\} (\subset \Sigma_{\text{target}})$ can occur in the ambient spacetime $\mathbb{M}^{D_{\text{amb}}}$ as a motion of the space 3-brane. Since this decrease in “set of possible motions” from Σ_{target} to $\Sigma_{\text{B}\cap\text{T}}^{(\text{ovlp})\cdot\text{PE}} (\subset \Sigma_{\text{target}})$ is similarly found for constitutive equations (e.g., Ohm’s law), the PE action $S_{\text{met}}^{(\text{PE})}[g_{\mu\nu}]$ making the overlap $\Sigma_{\text{B}\cap\text{T}}^{(\text{ovlp})\cdot\text{PE}}$ produces a “constitutive equation” specific to the space 3-brane in $\mathbb{M}^{D_{\text{amb}}}$.

Our AAT method using the metric action $S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}]$ seems similarly found in General Relativity: in General Relativity, the locally inertial coordinates (LIC) $\xi^{\hat{\alpha}}$ are studied by solving the PDE $\partial_\mu \xi^{\hat{\alpha}} \partial_\nu \xi^{\hat{\beta}} \eta_{\hat{\alpha}\hat{\beta}} = \mathbf{g}_{\mu\nu}^{\text{sol}}$ with $(\delta S_{\text{EH}}/\delta \mathbf{g}_{\mu\nu})[\mathbf{g}_{\mu\nu}^{\text{sol}}] = 0$ (this situation corresponds to Eqs. (5.8) and (5.9)). Moreover, these LIC $\xi^{\hat{\alpha}}$ are similar to the embedding f^A of the action $S_{\text{emb}}^{(3\text{br})}[f^A]$, because (i) $\xi^{\hat{\alpha}}$ appear in the “GR metric” $\mathbf{g}_{\mu\nu} = \partial_\mu \xi^{\hat{\alpha}} \partial_\nu \xi^{\hat{\beta}} \eta_{\hat{\alpha}\hat{\beta}}$ like f^A in $\gamma_{\mu\nu} = \partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}}$, and (ii) $\xi^{\hat{\alpha}}$ form an immersion of x^μ like f^A due to $\text{rank}(\partial_\mu \xi^{\hat{\alpha}}) = \text{rank}(\partial_\mu f^A) = 4$, where $\partial_\mu \xi^{\hat{\alpha}}$ is the vierbein. Then, due to these *similarities* between $\xi^{\hat{\alpha}}$ and f^A , the analogical reasoning can support that the embedding f^A has the metric action $S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}]$ like the LIC $\xi^{\hat{\alpha}}$ having $S_{\text{EH}}[\mathbf{g}_{\mu\nu}]$. For the use of these actions $S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}]$ and $S_{\text{EH}}[\mathbf{g}_{\mu\nu}]$, see between Eqs. (5.14) and (5.24).

Mathematically, the AAT method using the overlapping action $S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}]$ consists of two main steps: (i) finding a solution $g_{\mu\nu}^{\text{sol}}$ of $(\delta S_{\text{met}}^{(\text{ovlp})}/\delta g_{\mu\nu})[g_{\mu\nu}] = 0$ as in Eq. (5.8), and next (ii) finding a solution f_{bul}^A of the rifle PDE $\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}} = g_{\mu\nu}^{\text{sol}}$ as in Eq. (5.9).

Since this “two-step AAT method” is a method for solving the two coupled equations $(\delta S_{\text{met}}^{(\text{ovlp})}/\delta g_{\mu\nu})[g_{\mu\nu}] = 0$ and $\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}} = g_{\mu\nu}$, we can try a different method, i.e., the insertion of the latter equation $\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}} = g_{\mu\nu}$ into the former

$$\left. \frac{\delta S_{\text{met}}^{(\text{ovlp})}}{\delta g_{\mu\nu}} \right|_{\text{repl}} \stackrel{\text{def}}{=} \frac{\delta S_{\text{met}}^{(\text{ovlp})}}{\delta g_{\mu\nu}} [\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}}] = 0, \quad (5.14)$$

where the symbol $|_{\text{repl}}$ denotes the replacement $g_{\mu\nu} \Rightarrow \partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}}$. This new equation $\delta S_{\text{met}}^{(\text{ovlp})}/\delta g_{\mu\nu}|_{\text{repl}} = 0$ is called the “replaced-equation” (cf. Eqs. (7.14) and (7.21)).

Because solving Eq. (5.14) is the same as solving the rifle PDE $\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}} = g_{\mu\nu}^{\text{sol}}$ of the two-step AAT method, the solution set of Eq. (5.14)

$$\Sigma_{\text{ovlp}}^{(\text{sol})} \stackrel{\text{def}}{=} \{ f^A \mid \delta S_{\text{met}}^{(\text{ovlp})} / \delta g_{\mu\nu} |_{\text{repl}} = 0 \} \quad (5.15)$$

is equal to the bullet set $\Sigma_{\text{bullet}}^{(\text{ovlp})}$ of the overlapping action $S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}]$, namely,

$$\Sigma_{\text{ovlp}}^{(\text{sol})} = \Sigma_{\text{bullet}}^{(\text{ovlp})} . \quad (5.16)$$

In Eq. (5.14), the replacement $|_{\text{repl}}$ was applied after the functional derivative $\delta/\delta g_{\mu\nu}$. Here, we apply the replacement $|_{\text{repl}}$ before the derivative $\delta/\delta g_{\mu\nu}$. This produces a new *functional* of the embedding f^A

$$\tilde{S}_{\text{met}}^{(\text{ovlp})}[f^A] \stackrel{\text{def}}{=} S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}]|_{\text{repl}} = S_{\text{met}}^{(\text{ovlp})}[\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}}] , \quad (5.17)$$

implying the Lagrangian $\tilde{\mathcal{L}}_{\text{met}}^{(\text{ovlp})}$ of this new functional $\tilde{S}_{\text{met}}^{(\text{ovlp})}[f^A]$ satisfies (cf. Eq. (4.27))

$$\tilde{\mathcal{L}}_{\text{met}}^{(\text{ovlp})}(\partial_\mu f^A, \dots) \stackrel{\text{def}}{=} \mathcal{L}_{\text{met}}^{(\text{ovlp})}(g_{\mu\nu}, \dots)|_{\text{repl}} . \quad (5.18)$$

Of course, it is possible that the original action $S_{\text{emb}}^{(3\text{br})}[f^A]$ in Eq. (4.27) takes the form of the new functional $\tilde{S}_{\text{met}}^{(\text{ovlp})}[f^A] = S_{\text{met}}^{(\text{ovlp})}[\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}}]$ (see Eq. (5.25)).

After the replacement in Eq. (5.17)

$$g_{\mu\nu} = \partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}} , \quad (5.19)$$

the functional derivative $\delta/\delta g_{\mu\nu}$ in Eq. (5.14) is replaced with $\delta/\delta(\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}})$. Thus, $\frac{\delta S_{\text{met}}^{(\text{ovlp})}}{\delta g_{\mu\nu}}|_{\text{repl}} = 0$ in Eq. (5.14) is expressed as

$$\frac{\delta \tilde{S}_{\text{met}}^{(\text{ovlp})}}{\delta(\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}})}[f^A] = 0 , \quad (5.20)$$

which is different from the *usual* variational equation

$$\frac{\delta \tilde{S}_{\text{met}}^{(\text{ovlp})}}{\delta f^A}[f^A] = 0 . \quad (5.21)$$

Since $\frac{\delta \tilde{S}_{\text{met}}^{(\text{ovlp})}}{\delta(\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}})} = 0$ in Eq. (5.20) is the same as $\frac{\delta S_{\text{met}}^{(\text{ovlp})}}{\delta g_{\mu\nu}}|_{\text{repl}} = 0$ in Eq. (5.14), the solution set $\tilde{\Sigma}_{\text{ovlp}}^{(\text{sol})}$ of the former equation

$$\tilde{\Sigma}_{\text{ovlp}}^{(\text{sol})} \stackrel{\text{def}}{=} \{ f^A \mid \delta \tilde{S}_{\text{met}}^{(\text{ovlp})} / \delta(\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}}) = 0 \} \quad (5.22)$$

satisfies

$$\tilde{\Sigma}_{\text{ovlp}}^{(\text{sol})} = \Sigma_{\text{ovlp}}^{(\text{sol})} = \Sigma_{\text{bullet}}^{(\text{ovlp})} \quad (\text{see Eq. (5.16)}) . \quad (5.23)$$

Due to the equality $\tilde{\Sigma}_{\text{ovlp}}^{(\text{sol})} = \Sigma_{\text{bullet}}^{(\text{ovlp})}$ in Eq. (5.23), the bullet $\Sigma_{\text{bullet}}^{(\text{ovlp})}$ is also produced by the odd-looking equation $\delta\tilde{S}_{\text{met}}^{(\text{ovlp})}/\delta(\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}}) = 0$ in Eqs. (5.20) and (5.22). Therefore, this equation $\delta\tilde{S}_{\text{met}}^{(\text{ovlp})}/\delta(\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}}) = 0$ can replace the equation of motion $\delta S_{\text{emb}}^{(3\text{br})}/\delta f^A = 0$ *within the overlapping B-T overlap* $\Sigma_{\text{B}\cap\text{T}}^{(\text{ovlp})} = \Sigma_{\text{bullet}}^{(\text{ovlp})} \cap \Sigma_{\text{target}} (\neq \emptyset)$ — the “equivalence” between these two equations within $\Sigma_{\text{B}\cap\text{T}}^{(\text{ovlp})}$. This conclusion becomes more evident, when we compare the bullet $\Sigma_{\text{bullet}}^{(\text{ovlp})} = \tilde{\Sigma}_{\text{ovlp}}^{(\text{sol})} = \{f^A | \delta\tilde{S}_{\text{met}}^{(\text{ovlp})}/\delta(\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}}) = 0\}$ with the target $\Sigma_{\text{target}} = \{f^A | \delta S_{\text{emb}}^{(3\text{br})}/\delta f^A = 0\}$ in Eq. (5.1).

In the above conclusion, we should be careful in interpreting the odd-looking equation $\delta\tilde{S}_{\text{met}}^{(\text{ovlp})}/\delta(\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}}) = 0$: this odd-looking equation must not be interpreted as the *equation of motion* for the space 3-brane, because (i) the space 3-brane already has its own equation of motion $\delta S_{\text{emb}}^{(3\text{br})}/\delta f^A = 0$, and (ii) the solution set $\tilde{\Sigma}_{\text{ovlp}}^{(\text{sol})} (= \Sigma_{\text{bullet}}^{(\text{ovlp})})$ in Eq. (5.22) may not be equal to the target Σ_{target} (see Eq. (5.28)). Then, the odd-looking equation $\delta\tilde{S}_{\text{met}}^{(\text{ovlp})}/\delta(\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}}) = 0$ may be interpreted, at best, as the constitutive equation specific to the space 3-brane (see above). Despite this, if we try to know the target Σ_{target} by using the new functional $\tilde{S}_{\text{met}}^{(\text{ovlp})}[f^A]$, the odd-looking equation $\delta\tilde{S}_{\text{met}}^{(\text{ovlp})}/\delta(\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}}) = 0$ should be used rather than the usual variational equation $\delta\tilde{S}_{\text{met}}^{(\text{ovlp})}/\delta f^A = 0$.

However, in General Relativity, it is well known that $(\delta S_{\text{EH}}/\delta \mathbf{g}_{\mu\nu})|_{\mathbf{g}_{\mu\nu} = \partial_\mu \xi^{\hat{\alpha}} \partial_\nu \xi^{\hat{\beta}} \eta_{\hat{\alpha}\hat{\beta}}} = 0$ *if and only if* $\delta\tilde{S}_{\text{EH}}/\delta(\partial_\mu \xi^{\hat{\alpha}}) = 0$, where $\partial_\mu \xi^{\hat{\alpha}}$ is the vierbein satisfying $\mathbf{g}_{\mu\nu} = \partial_\mu \xi^{\hat{\alpha}} \partial_\nu \xi^{\hat{\beta}} \eta_{\hat{\alpha}\hat{\beta}}$. (For the mathematical proof, see Ref. [2].) From the similarities of f^A to $\xi^{\hat{\alpha}}$ (see above), we easily confirm the equivalence that

$$\frac{\delta\tilde{S}_{\text{met}}^{(\text{ovlp})}}{\delta(\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}})}[f^A] = 0 \quad \text{if and only if} \quad \frac{\delta\tilde{S}_{\text{met}}^{(\text{ovlp})}}{\delta(\partial_\mu f^A)}[f^A] = 0. \quad (5.24)$$

For the special case of

$$S_{\text{emb}}^{(3\text{br})}[f^A] = \tilde{S}_{\text{met}}^{(\text{ovlp})}[f^A] = S_{\text{met}}^{(\text{ovlp})}[\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}}] \quad (\text{see Eq. (5.17)}) , \quad (5.25)$$

the usual variational equation $\delta\tilde{S}_{\text{met}}^{(\text{ovlp})}/\delta f^A = 0$ in Eq. (5.21) becomes the equation of motion for the space 3-brane. According to

$$\delta\tilde{S}_{\text{met}}^{(\text{ovlp})}/\delta f^A = \left[\delta\tilde{S}_{\text{met}}^{(\text{ovlp})}/\delta(\partial_\mu f^C \partial_\nu f^D \eta_{CD}^{\text{bulk}}) \right] \times \left[\delta(\partial_\mu f^M \partial_\nu f^N \eta_{MN}^{\text{bulk}})/\delta f^A \right] \quad (5.26)$$

$$\equiv -2 \partial_\mu \left\{ \left[\delta\tilde{S}_{\text{met}}^{(\text{ovlp})}/\delta(\partial_\mu f^C \partial_\nu f^D \eta_{CD}^{\text{bulk}}) \right] \partial_\nu f^M \eta_{AM}^{\text{bulk}} \right\} , \quad (5.27)$$

the odd-looking equation $\delta\tilde{S}_{\text{met}}^{(\text{ovlp})}/\delta(\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}}) = 0$ is not a necessary but *sufficient* condition for the equation of motion $\delta\tilde{S}_{\text{met}}^{(\text{ovlp})}/\delta f^A = 0$.

This means

$$\tilde{\Sigma}_{\text{ovlp}}^{(\text{sol})\cdot\text{sp}} \subsetneq \Sigma_{\text{target}}^{\text{sp}} \quad (\text{thus } \Sigma_{\text{B}\cap\text{T}}^{(\text{ovlp})\cdot\text{sp}} \subsetneq \Sigma_{\text{target}}^{\text{sp}}) , \quad (5.28)$$

where the superscript “sp” denotes the special case $S_{\text{emb}}^{(3\text{br})}[f^A] = \tilde{S}_{\text{met}}^{(\text{ovlp})}[f^A]$, and $\Sigma_{\text{target}}^{\text{sp}} \stackrel{\text{def}}{=} \{f_{\text{sol}}^A | (\delta\tilde{S}_{\text{met}}^{(\text{ovlp})}/\delta f^A)[f_{\text{sol}}^A] = 0\}$ is the target set for this special case. The inequality $\tilde{\Sigma}_{\text{ovlp}}^{(\text{sol})\cdot\text{sp}} \subsetneq$

Table 2: Symmetries in the Aim-At-Target (AAT) Method

ACTION	$ISO(1, D_{\text{amb}} - 1)$	Diff(4)
$S_{\text{emb}}^{(3\text{br})}[f^A]$	O	O
$S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}]$	O	O (X [†])

(O: preserving, X: breaking)

[†] The Diff(4) invariance may be broken by an ME action $S_{\text{met}}^{(\text{ME})}[g_{\mu\nu}]$, as said in the text.

$\Sigma_{\text{target}}^{\text{sp}}$ in Eq. (5.28) supports the above statement that $\delta\tilde{S}_{\text{met}}^{(\text{ovlp})}/\delta(\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}}) = 0$ must not be interpreted as the equation of motion for the space 3-brane.

Due to $\tilde{\Sigma}_{\text{ovlp}}^{(\text{sol})\cdot\text{sp}} \subsetneq \Sigma_{\text{target}}^{\text{sp}}$, the special case $S_{\text{emb}}^{(3\text{br})}[f^A] = \tilde{S}_{\text{met}}^{(\text{ovlp})}[f^A]$ in Eq. (5.25) does not have the “defect” of

$$\tilde{\Sigma}_{\text{ovlp}}^{(\text{sol})} - \Sigma_{\text{target}} \neq \emptyset, \quad (5.29)$$

which means that the bullet set $\tilde{\Sigma}_{\text{ovlp}}^{(\text{sol})}$ in Eq. (5.22) contains elements outside the target set $\Sigma_{\text{target}} = \{f_{\text{sol}}^A\}$. This suggests defining the “contained metric action” $S_{\text{met}}^{(\text{cont})}[g_{\mu\nu}]$ as an overlapping action $S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}]$ satisfying $\tilde{\Sigma}_{\text{ovlp}}^{(\text{sol})} \subsetneq \Sigma_{\text{target}}$ (i.e., without the defect of $\tilde{\Sigma}_{\text{ovlp}}^{(\text{sol})} - \Sigma_{\text{target}} \neq \emptyset$). The overlapping action $S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}]$ of the special case in Eq. (5.25) is an example of the contained action.

As implied in Eq. (5.29), for evaluating an ME action $S_{\text{met}}^{(\text{ME})}[g_{\mu\nu}]$, we may use the “bullet-target (B-T) difference”

$$\Delta_{\text{BT}} \stackrel{\text{def}}{=} \Sigma_{\text{bullet}} - \Sigma_{\text{target}} \quad (5.30)$$

together with the B-T overlap $\Sigma_{\text{B}\cap\text{T}} = \Sigma_{\text{bullet}} \cap \Sigma_{\text{target}}$ in Eq. (5.12). For example, a “large” B-T overlap $\Sigma_{\text{B}\cap\text{T}}$ and a “small” B-T difference Δ_{BT} can result in a “good” ME action $S_{\text{met}}^{(\text{ME})}[g_{\mu\nu}]$.

6 The Symmetries and the Forms of the Overlapping Metric Action in the AAT Method

Now, in terms of symmetries, we study the *forms* of the overlapping metric action $S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}]$ (see Table 2). The B-T overlap $\Sigma_{\text{B}\cap\text{T}}^{(\text{ovlp})} = \Sigma_{\text{bullet}}^{(\text{ovlp})} \cap \Sigma_{\text{target}}$ represents the maximum knowledge which we can obtain about the target $\Sigma_{\text{target}} = \{f_{\text{sol}}^A\}$ of the original action $S_{\text{emb}}^{(3\text{br})}[f^A]$ by using the chosen metric action $S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}]$.

First, we consider the symmetries of the target $\Sigma_{\text{target}} = \{f_{\text{sol}}^A\}$ of the original action $S_{\text{emb}}^{(3\text{br})}[f^A]$, as follows: since the original action $S_{\text{emb}}^{(3\text{br})}[f^A]$ is invariant under $ISO(1, D_{\text{amb}} - 1)$

and $\text{Diff}(4)$, the definition of the invariance of this action $S_{\text{emb}}^{(3\text{br})}[f^A]$ implies that the solution set $\Sigma_{\text{target}} = \{f_{\text{sol}}^A\}$ of the action $S_{\text{emb}}^{(3\text{br})}[f^A]$ is also invariant under $ISO(1, D_{\text{amb}} - 1)$ and $\text{Diff}(4)$ (see Sec. 4). By definition, the induced-metric set $\Sigma_{\text{ind}} = \{\gamma_{\mu\nu}\}$ in Eq. (5.3) is invariant under $ISO(1, D_{\text{amb}} - 1)$ and $\text{Diff}(4)$. Note $\gamma'_{\mu\nu} = \gamma_{\mu\nu}$ under $ISO(1, D_{\text{amb}} - 1)$, and $\gamma'_{\rho\sigma} = \frac{\partial x^\mu}{\partial x'^\rho} \frac{\partial x^\nu}{\partial x'^\sigma} \gamma_{\mu\nu}$ under $\text{Diff}(4)$.

Next, we consider the symmetries of the bullet $\Sigma_{\text{bullet}}^{(\text{ovlp})} = \{f_{\text{bul}}^A\}$ of the overlapping action $S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}]$, as follows: since $g_{\mu\nu} = \partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}}$ in Eq. (5.19) is already invariant under $ISO(1, D_{\text{amb}} - 1)$, the action $S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}]$ has the $ISO(1, D_{\text{amb}} - 1)$ invariance. This implies its solution-metric set $\Sigma_{\text{met}}^{(\text{sol})\cdot\text{ovlp}} = \{g_{\mu\nu}^{\text{sol}}\}$ also has the $ISO(1, D_{\text{amb}} - 1)$ invariance.

Then, since both $g_{\mu\nu}^{\text{sol}}$ and $\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}}$ are invariant under $ISO(1, D_{\text{amb}} - 1)$, $\Sigma_{\text{bullet}}^{(\text{ovlp})} \ni f'^A = \Lambda_B^A f^B + c^A$ is equivalent to $\Sigma_{\text{bullet}}^{(\text{ovlp})} \ni f^A$, which means the bullet $\Sigma_{\text{bullet}}^{(\text{ovlp})} = \{f_{\text{bul}}^A\}$ has the $ISO(1, D_{\text{amb}} - 1)$ invariance like the target $\Sigma_{\text{target}} = \{f_{\text{sol}}^A\}$. Therefore, since the intersection of two G-invariant sets is G-invariant (G: a group), the B-T overlap $\Sigma_{\text{B}\cap\text{T}}^{(\text{ovlp})} = \Sigma_{\text{bullet}}^{(\text{ovlp})} \cap \Sigma_{\text{target}}$ is invariant under $ISO(1, D_{\text{amb}} - 1)$.

Before studying the $\text{Diff}(4)$ symmetry properties, we need to re-consider the (i) mathematical and (ii) physical existences of the overlapping action $S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}]$ (see the summary of the AAT method in Sec. 5):

First, we study an ME action $S_{\text{met}}^{(\text{ME})}[g_{\mu\nu}]$, which is a mere tool for knowing the target Σ_{target} . Since this mere tool $S_{\text{met}}^{(\text{ME})}[g_{\mu\nu}]$ cannot forbid any element f_{sol}^A of the target set $\Sigma_{\text{target}} = \{f_{\text{sol}}^A\}$, we can use a $\text{Diff}(4)$ -breaking or a $\text{Diff}(4)$ -preserving action $S_{\text{met}}^{(\text{ME})}[g_{\mu\nu}]$ as long as this ME action $S_{\text{met}}^{(\text{ME})}[g_{\mu\nu}]$ provides a considerable information about the target Σ_{target} .

For example, we can use a $\text{Diff}(4)$ -breaking ME action $S_{\text{met}}^{(\text{ME})}[g_{\mu\nu}]$ as a mere tool for Σ_{target} . Since this action $S_{\text{met}}^{(\text{ME})}[g_{\mu\nu}]$ is not invariant under $\text{Diff}(4)$ unlike the original action $S_{\text{emb}}^{(3\text{br})}[f^A]$, the solution set $\Sigma_{\text{met}}^{(\text{sol})\cdot\text{ME}}$ of the metric action $S_{\text{met}}^{(\text{ME})}[g_{\mu\nu}]$ has an element $g_{\mu\nu}^{\text{sol}\sharp}$ satisfying

$$g_{\mu\nu}^{\text{sol}\sharp} \in \Sigma_{\text{met}}^{(\text{sol})\cdot\text{ME}} \quad \text{but} \quad g_{\rho\sigma}^{\text{sol}\sharp'} \notin \Sigma_{\text{met}}^{(\text{sol})\cdot\text{ME}} \quad \text{for an element } \Phi_{4\text{D}}^\sharp \text{ of } \text{Diff}(4), \quad (6.1)$$

where $g_{\rho\sigma}^{\text{sol}\sharp'} = \frac{\partial x^\mu}{\partial x'^\rho} \frac{\partial x^\nu}{\partial x'^\sigma} g_{\mu\nu}^{\text{sol}\sharp}$ with $x' = \Phi_{4\text{D}}^\sharp(x)$. This means the solution-metric set $\Sigma_{\text{met}}^{(\text{sol})\cdot\text{ME}} = \{g_{\mu\nu}^{\text{sol}\cdot\text{ME}}\}$ breaks the $\text{Diff}(4)$ invariance.

Despite this, if the $\text{Diff}(4)$ -breaking set $\Sigma_{\text{met}}^{(\text{sol})\cdot\text{ME}} = \{g_{\mu\nu}^{\text{sol}\cdot\text{ME}}\}$ contains a “ $\text{Diff}(4)$ gauge slice” $\Sigma_{\text{ind}}^{(\text{GS})}$ of the $\text{Diff}(4)$ -preserving set $\Sigma_{\text{ind}} = \{\gamma_{\mu\nu}\}$, we can still know the target $\Sigma_{\text{target}} = \{f_{\text{sol}}^A\}$ by, for example, (i) finding a solution f_{sol}^A ($\in \Sigma_{\text{target}}$) of the rifle PDE $\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}} = g_{\mu\nu}^{\text{sol}\cdot\text{ME}} \in \Sigma_{\text{ind}}^{(\text{GS})}$, and (ii) applying $\text{Diff}(4)$ to this solution f_{sol}^A , which forms its “ $\text{Diff}(4)$ gauge orbit” $\langle f_{\text{sol}}^A \rangle_{\text{diff}}$. This aspect is similarly found in a gauge theory, where the gauge invariance is broken by adding a gauge-fixing term.

Thus, the breaking of the $\text{Diff}(4)$ invariance by the ME action $S_{\text{met}}^{(\text{ME})}[g_{\mu\nu}]$ may not be a serious problem for knowing the target Σ_{target} (see the symbol X in Table 2). Of course, we can use a $\text{Diff}(4)$ -preserving ME action $S_{\text{met}}^{(\text{ME})}[g_{\mu\nu}]$ as another mere tool for the target Σ_{target} .

Next, we study a PE action $S_{\text{met}}^{(\text{PE})}[g_{\mu\nu}]$, which has the physical existence unlike the ME action $S_{\text{met}}^{(\text{ME})}[g_{\mu\nu}]$. Then, since the PE action $S_{\text{met}}^{(\text{PE})}[g_{\mu\nu}]$ produces the constitutive equation, only the element f_{ove}^A of the B-T overlap $\Sigma_{\text{B}\cap\text{T}}^{(\text{PE})} = \{f_{\text{ove}}^A\}$ ($\subset \Sigma_{\text{target}}$) can be a motion of the space 3-brane in $\mathbb{M}^{D_{\text{amb}}}$, as said in Sec. 5.

The PE action $S_{\text{met}}^{(\text{PE})}[g_{\mu\nu}]$ can determine the $\text{Diff}(4)$ symmetry property of the B-T overlap $\Sigma_{\text{B}\cap\text{T}}^{(\text{PE})} = \Sigma_{\text{bullet}}^{(\text{PE})} \cap \Sigma_{\text{target}}$ through its bullet $\Sigma_{\text{bullet}}^{(\text{PE})}$: for example, we consider a $\text{Diff}(4)$ -breaking PE action $S_{\text{met}}^{(\text{PE})}[g_{\mu\nu}]$, whose solution-metric set $\Sigma_{\text{met}}^{(\text{sol})\cdot\text{PE}}$ has an element $g_{\mu\nu}^{\text{sol}\sharp}$ satisfying

$$g_{\mu\nu}^{\text{sol}\sharp} \in \Sigma_{\text{met}}^{(\text{sol})\cdot\text{PE}} \quad \text{but} \quad g_{\rho\sigma}^{\text{sol}\sharp'} \notin \Sigma_{\text{met}}^{(\text{sol})\cdot\text{PE}} \quad \text{for an element } \Phi_{4\text{D}}^{\sharp} \text{ of } \text{Diff}(4), \quad (6.2)$$

where $g_{\rho\sigma}^{\text{sol}\sharp'} = \frac{\partial x^\mu}{\partial x'^\rho} \frac{\partial x^\nu}{\partial x'^\sigma} g_{\mu\nu}^{\text{sol}\sharp}$ with $x' = \Phi_{4\text{D}}^{\sharp}(x)$. This means the breaking of the $\text{Diff}(4)$ invariance by the solution-metric set $\Sigma_{\text{met}}^{(\text{sol})\cdot\text{PE}} = \{g_{\mu\nu}^{\text{sol}\cdot\text{PE}}\}$.

Suppose that a 4DL solution $f_{\text{sol}\sharp}^A$ ($\in \Sigma_{\text{target}}$) of the original action $S_{\text{emb}}^{(3\text{br})}[f^A]$ satisfies

$$\partial_\mu f_{\text{sol}\sharp}^A \partial_\nu f_{\text{sol}\sharp}^B \eta_{AB}^{\text{bulk}} = g_{\mu\nu}^{\text{sol}\sharp} \quad (\text{i.e., } f_{\text{sol}\sharp}^A \in \Sigma_{\text{bullet}}^{(\text{PE})}), \quad (6.3)$$

which means the induced metric $\gamma_{\mu\nu}^{\sharp} = \partial_\mu f_{\text{sol}\sharp}^A \partial_\nu f_{\text{sol}\sharp}^B \eta_{AB}^{\text{bulk}}$ (cf. Eq. (4.10)) has the equality

$$\gamma_{\mu\nu}^{\sharp} = g_{\mu\nu}^{\text{sol}\sharp}. \quad (6.4)$$

Then, the transformed 4DL solution $f_{\text{sol}\sharp}^A(x') = f_{\text{sol}\sharp}^A(x)$ with $x' = \Phi_{4\text{D}}^{\sharp}(x)$ satisfies

$$\partial'_\rho f_{\text{sol}\sharp}^A \partial'_\sigma f_{\text{sol}\sharp}^B \eta_{AB}^{\text{bulk}} = g_{\rho\sigma}^{\text{sol}\sharp'}, \quad (6.5)$$

where $f_{\text{sol}\sharp}^A$ ($\in \Sigma_{\text{target}}$) is an element of the $\text{Diff}(4)$ gauge orbit $\langle f_{\text{sol}\sharp}^A \rangle_{\text{diff}}$.

Due to $g_{\rho\sigma}^{\text{sol}\sharp'} \notin \Sigma_{\text{met}}^{(\text{sol})\cdot\text{PE}}$ in Eq. (6.2), the transformed 4DL solution $f_{\text{sol}\sharp}^A$ ($\in \Sigma_{\text{target}}$) does not belong to the bullet $\Sigma_{\text{bullet}}^{(\text{PE})}$ (i.e., $f_{\text{sol}\sharp}^A \notin \Sigma_{\text{bullet}}^{(\text{PE})}$) unlike the original 4DL solution $f_{\text{sol}\sharp}^A$ in Eq. (6.3). Thus, like the bullet $\Sigma_{\text{bullet}}^{(\text{PE})}$, the B-T overlap $\Sigma_{\text{B}\cap\text{T}}^{(\text{PE})}$ describing the space 3-brane breaks the $\text{Diff}(4)$ invariance, because

$$\Sigma_{\text{B}\cap\text{T}}^{(\text{PE})} \ni f_{\text{sol}\sharp}^A \quad \text{but} \quad \Sigma_{\text{B}\cap\text{T}}^{(\text{PE})} \not\ni f_{\text{sol}\sharp}^A. \quad (6.6)$$

To sum up, the $\text{Diff}(4)$ -breaking PE action $S_{\text{met}}^{(\text{PE})}[g_{\mu\nu}]$ may imply the $\text{Diff}(4)$ -breaking B-T overlap $\Sigma_{\text{B}\cap\text{T}}^{(\text{PE})}$.

However, the $\text{Diff}(4)$ -breaking B-T overlap $\Sigma_{\text{B}\cap\text{T}}^{(\text{PE})}$ can cause a physical problem of being contrary to the observed General Relativity: since only the element f_{ove}^A of the B-T overlap $\Sigma_{\text{B}\cap\text{T}}^{(\text{PE})} = \{f_{\text{ove}}^A\}$ can occur in the ambient spacetime $\mathbb{M}^{D_{\text{amb}}}$ as a motion of the space 3-brane (see Sec. 5), the latter result $\Sigma_{\text{B}\cap\text{T}}^{(\text{PE})} \not\ni f_{\text{sol}\sharp}^A$ in Eq. (6.6) forbids $\gamma_{\rho\sigma}^{\sharp'} = \partial'_\rho f_{\text{sol}\sharp}^A \partial'_\sigma f_{\text{sol}\sharp}^B \eta_{AB}^{\text{bulk}}$ to occur in $\mathbb{M}^{D_{\text{amb}}}$ *unlike* the former $\Sigma_{\text{B}\cap\text{T}}^{(\text{PE})} \ni f_{\text{sol}\sharp}^A$, which allows $\gamma_{\mu\nu}^{\sharp} = \partial_\mu f_{\text{sol}\sharp}^A \partial_\nu f_{\text{sol}\sharp}^B \eta_{AB}^{\text{bulk}}$ to occur in $\mathbb{M}^{D_{\text{amb}}}$. Thus, due to the approximation $\mathbf{g}_{\mu\nu} \approx \gamma_{\mu\nu}$ in Eq. (4.14), $\mathbf{g}_{\rho\sigma}^{\sharp'} (\approx \gamma_{\rho\sigma}^{\sharp'})$ cannot

occur in $\mathbb{M}^{D_{\text{amb}}}$ unlike $\mathbf{g}_{\mu\nu}^\#$ ($\approx \gamma_{\mu\nu}^\#$). This means that the primed GR metric $\mathbf{g}_{\rho\sigma}^{\#t}$ cannot be a solution of General Relativity unlike the unprimed one $\mathbf{g}_{\mu\nu}^\#$. As a result, General Relativity should be a Diff(4)-breaking theory, which is falsified by observations.

Therefore, it is natural to use only a Diff(4)-preserving PE action $S_{\text{met}}^{(\text{PE})}[g_{\mu\nu}]$ which produces the Diff(4)-preserving B-T overlap $\Sigma_{\text{B}\cap\text{T}}^{(\text{PE})}$. In addition, since it is not compulsory that the ME action $S_{\text{met}}^{(\text{ME})}[g_{\mu\nu}]$ breaks the Diff(4) invariance, we can choose to use a Diff(4)-preserving ME action $S_{\text{met}}^{(\text{ME})}[g_{\mu\nu}]$. To sum up, we use only a Diff(4)-invariant case of the overlapping action $S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}]$, irrespective of whether it is an ME or PE action.

For a Diff(4)-invariant overlapping action $S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}]$, due to the definition $\Sigma_{\text{B}\cap\text{T}}^{(\text{ovlp})} \stackrel{\text{def}}{=} \Sigma_{\text{bullet}}^{(\text{ovlp})} \cap \Sigma_{\text{target}}$, every element f_{ove}^A of the B-T overlap $\Sigma_{\text{B}\cap\text{T}}^{(\text{ovlp})} = \{f_{\text{ove}}^A\}$ satisfies

$$f_{\text{ove}}^A = f_{\text{bul}}^A = f_{\text{sol}}^A \quad \text{with } f_{\text{bul}}^A \in \Sigma_{\text{bullet}}^{(\text{ovlp})} \text{ and } f_{\text{sol}}^A \in \Sigma_{\text{target}} , \quad (6.7)$$

which results in

$$\partial_\mu f_{\text{ove}}^A \partial_\nu f_{\text{ove}}^B \eta_{AB}^{\text{bulk}} = g_{\mu\nu}^{\text{sol}} = \gamma_{\mu\nu} . \quad (6.8)$$

From Eq. (6.8), we obtain the equality for the solution metric

$$g_{\mu\nu}^{\text{sol}} = \gamma_{\mu\nu} \quad \text{for every element } f_{\text{ove}}^A \text{ of } \Sigma_{\text{B}\cap\text{T}}^{(\text{ovlp})} \quad (\text{cf. Eq. (4.14)}) . \quad (6.9)$$

This equality $g_{\mu\nu}^{\text{sol}} = \gamma_{\mu\nu}$ within the B-T overlap $\Sigma_{\text{B}\cap\text{T}}^{(\text{ovlp})}$ means that, below the “metric cutoff” Λ_{met} , the metric $g_{\mu\nu}$ can describe the “emergent field” $\gamma_{\mu\nu}$ ($= \partial_\mu f_{\text{sol}}^A \partial_\nu f_{\text{sol}}^B \eta_{AB}^{\text{bulk}}$), which is derived from the locations ($\in \mathbb{M}^{D_{\text{amb}}}$) of space quanta occupying $\mathbb{M}^{D_{\text{amb}}}$.

Moreover, due to the equality $g_{\mu\nu}^{\text{sol}} = \gamma_{\mu\nu}$, the spacetime $\mathcal{S}_{\text{met}}^{4\text{D}}$ having this metric $g_{\mu\nu}^{\text{sol}}$ is exactly the same as the world volume \mathcal{WV}_{sq} of the space 3-brane, i.e.,

$$\mathcal{S}_{\text{met}}^{4\text{D}} = \mathcal{WV}_{\text{sq}} \quad \text{for every element } f_{\text{ove}}^A \text{ of } \Sigma_{\text{B}\cap\text{T}}^{(\text{ovlp})} \quad (\text{cf. Eq. (4.13)}) . \quad (6.10)$$

This equality $\mathcal{S}_{\text{met}}^{4\text{D}} = \mathcal{WV}_{\text{sq}}$ within the B-T overlap $\Sigma_{\text{B}\cap\text{T}}^{(\text{ovlp})}$ means that, below the cutoff Λ_{met} , the spacetime $\mathcal{S}_{\text{met}}^{4\text{D}}$ with the metric $g_{\mu\nu}^{\text{sol}}$ can describe the “emergent spacetime” \mathcal{WV}_{sq} , which is formed by the world lines \mathcal{WL}_{sq} ($\subset \mathbb{M}^{D_{\text{amb}}}$) of many space quanta in $\mathbb{M}^{D_{\text{amb}}}$.

In sum, by Eqs. (6.9) and (6.10), we have the equality for the two spacetime manifolds

$$(\mathcal{S}_{\text{met}}^{4\text{D}}, g_{\mu\nu}^{\text{sol}}) = (\mathcal{WV}_{\text{sq}}, \gamma_{\mu\nu}) \quad \text{within } \Sigma_{\text{B}\cap\text{T}}^{(\text{ovlp})} \text{ (below } \Lambda_{\text{met}}) . \quad (6.11)$$

Exact values for the spacetime measurements are provided by the exact or true spacetime $(\mathcal{WV}_{\text{sq}}, \gamma_{\mu\nu})$, which is the 4D emergent spacetime occupying the ambient spacetime $\mathbb{M}^{D_{\text{amb}}}$.

Now, for the Diff(4)-preserving overlapping action $S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}]$, we consider the form of its Lagrangian $\mathcal{L}_{\text{met}}^{(\text{ovlp})}(\Lambda_{\text{met}}; g_{\mu\nu})$ more closely: as in usual effective theories, this Lagrangian $\mathcal{L}_{\text{met}}^{(\text{ovlp})}$ having its own UV cutoff Λ_{met} (cf. Eq. (5.6)) can be expressed as

$$\mathcal{L}_{\text{met}}^{(\text{ovlp})}(\Lambda_{\text{met}}; g_{\mu\nu}) = \sum c_k \frac{\mathcal{O}_k}{\Lambda_{\text{met}}^{d_k-4}} , \quad (6.12)$$

where the coefficient c_k has no mass dimension, and the local operator \mathcal{O}_k of mass dimension d_k consists of the metric $g_{\mu\nu}$ and its derivatives. To make $\mathcal{L}_{\text{met}}^{(\text{ovlp})}(\Lambda_{\text{met}}; g_{\mu\nu})$ invariant under $\text{Diff}(4)$, we assume every operator \mathcal{O}_k is invariant under $\text{Diff}(4)$. Since the $\text{Diff}(4)$ invariance of the overlapping action $S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}]$ is shared by General Relativity, we easily expect this metric action $S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}]$ to contain the Einstein-Hilbert action (see Eq. (6.17)).

Generally speaking, since the metric Lagrangian $\mathcal{L}_{\text{met}}^{(\text{ovlp})}$ having the derivatives of $g_{\mu\nu}$ can contain at least one dimensionful parameter (say, ξ_{met}) to maintain its mass dimension $[\mathcal{L}_{\text{met}}^{(\text{ovlp})}] = 4$, the Lagrangian $\mathcal{L}_{\text{met}}^{(\text{ovlp})}$ becomes the function of the parameter ξ_{met} , which has a Laurent series for ξ_{met} . Thus, this Laurent series with $\xi_{\text{met}} = \Lambda_{\text{met}}$ can lead to the series like Eq. (6.12), even when the effective-theory nature of $\mathcal{L}_{\text{met}}^{(\text{ovlp})}(\Lambda_{\text{met}}; g_{\mu\nu})$ is not considered.

If we (i) observe at an energy $E_{\text{obs}} (\lesssim \Lambda_{\text{met}})$, and (ii) neglect all the operators with $d_k \geq d_{\text{negl}}$, then the error $\varepsilon_{\text{negl}}$ has a size of $O(E_{\text{obs}}/\Lambda_{\text{met}})^{d_{\text{negl}}-4}$, implying

$$d_{\text{negl}} \approx 4 + \frac{\log \varepsilon_{\text{negl}}}{\log(E_{\text{obs}}/\Lambda_{\text{met}})} . \quad (6.13)$$

This leads to the approximate predictive power that a computation with the error $\varepsilon_{\text{negl}}$ requires only a finite number of operators \mathcal{O}_k up to the maximally allowed mass dimension $d_{\text{max}} (< d_{\text{negl}})$.

When the operator \mathcal{O}_k in Eq. (6.12) contains N_{∂} derivatives ∂_{α} and N_g metrics $g_{\mu\nu}$, the $\text{Diff}(4)$ invariance requires the operator \mathcal{O}_k to possess $\frac{1}{2}N_{\partial} + N_g$ inverse metrics $g^{\mu\nu}$ for contraction. The mass dimension of \mathcal{O}_k satisfies

$$d_k = [\mathcal{O}_k] = [(\hat{g}^{-1})^{\frac{1}{2}N_{\partial}+N_g} \times \partial^{N_{\partial}} \times \hat{g}^{N_g}] = [(\hat{g} dX dX)^{-\frac{1}{2}N_{\partial}}] = N_{\partial} , \quad (6.14)$$

where the four symbols have correspondences $\hat{g}^{-1} \leftrightarrow g^{\mu\nu}$, $\partial \leftrightarrow \partial_{\mu}$, $\hat{g} \leftrightarrow g_{\mu\nu}$ and $dX \leftrightarrow dx^{\mu}$.

Since the number $\frac{1}{2}N_{\partial} + N_g$ of inverse metrics $g^{\mu\nu}$ should be an integer (≥ 0),

$$N_{\partial} = 2 \times (\text{integer}) , \quad (6.15)$$

implying d_k is an *even* integer due to $d_k = N_{\partial}$ in Eq. (6.14). Thus, the “overlapping Lagrangian” $\mathcal{L}_{\text{met}}^{(\text{ovlp})}(\Lambda_{\text{met}}; g_{\mu\nu})$ in Eq. (6.12) has the derivative expansion

$$\mathcal{L}_{\text{met}}^{(\text{ovlp})}(\Lambda_{\text{met}}; g_{\mu\nu}) = \sum_{d_k: \text{even}} c_{d_k} \Lambda_{\text{met}}^4 O\left(\frac{\partial}{\Lambda_{\text{met}}}\right)^{d_k} , \quad (6.16)$$

where d_k are non-negative even integers.

The overlapping Lagrangian $\mathcal{L}_{\text{met}}^{(\text{ovlp})}(\Lambda_{\text{met}}; g_{\mu\nu})$ in Eq. (6.16) can have the form of

$$\mathcal{L}_{\text{met}}^{(\text{ovlp})} = c_0 \Lambda_{\text{met}}^4 + c_2 \Lambda_{\text{met}}^2 R + c_4^{(1)} R^2 + c_4^{(2)} R_{\mu\nu} R^{\mu\nu} + c_4^{(3)} g^{\mu\nu} \nabla_{\mu} R \nabla_{\nu} R + \dots , \quad (6.17)$$

where all the coefficients (e.g., c_0, c_2) are dimensionless, and both of the covariant derivative ∇_{μ} and the curvature quantities (e.g., R) are built from the metric $g_{\mu\nu}$ (see Ref. [3]).

7 The Effective Theory for the Universe: the Inclusion of Matter

According to observations, our universe contains various particles (e.g., leptons) which are different in kind from space quanta. To distinguish those particles from the space quanta, we coin a new term **occupant quantum** (OQ) denoting any particle which (i) differs from space quanta, and (ii) occupies the space 3-brane without departing from it (i.e., the confinement of the occupant quantum to the space 3-brane).

To sum up, our universe can be regarded as a *composite system* which consists of space quanta and occupant quanta, moving within the ambient spacetime $\mathbb{M}^{D_{\text{amb}}}$.

Since space quantum is more fundamental than graviton, there can be a scenario that every particle of the Standard Model (SM) is a bound state of occupant quanta. However, there can be another scenario that each SM particle is identified with a single occupant quantum. Besides these, there can be various other scenarios.

Despite this, from now on, we will consider only the *low-energy* spectrum (e.g., the SM particles) of occupant quanta which can be observed at low enough energies: since each of these observable occupant quanta is confined to the world volume \mathcal{WV}_{sq} of the space 3-brane, it is described by a function Ψ_{OQ} whose domain is the world volume \mathcal{WV}_{sq} . For a brane-chart x^μ of \mathcal{WV}_{sq} , the “brane-field” Ψ_{OQ} on \mathcal{WV}_{sq} is represented as the function $\Psi_{\text{OQ}}(x^\mu)$ of the four coordinates x^μ .

The *value* $\Psi_{\text{OQ}}(x^\mu(p))$ at a point $p \in \mathcal{WV}_{\text{sq}}$ is either (i) a “brane-tensor of a type” (e.g., a scalar) of \mathcal{WV}_{sq} , or (ii) a “brane-spinor” (e.g., a Weyl spinor) of the “brane Lorentz group” $SO(1,3)$ at the point p . The vierbein e_μ^a satisfying $e_\mu^a e_\nu^b \eta_{ab} = \gamma_{\mu\nu}$ can be used in the action for brane-spinors. Suppose that the bosons and fermions of the Standard Model are described by their corresponding brane-fields $\Psi_{\text{OQ}}^{(\text{SM})}$. Like the induced metric $\gamma_{\mu\nu}(x)$, all the SM brane-fields $\Psi_{\text{OQ}}^{(\text{SM})}(x)$ are invariant (i.e., “bulk-scalars”) under every $B \Rightarrow B'$ transformation $Y^A \rightarrow Y'^A \in ISO(1, D_{\text{amb}} - 1)$ between the bulk-charts Y^A and Y'^A of $\mathbb{M}^{D_{\text{amb}}}$.

The action $S_{\text{OQ}}^{(3\text{br})}$ for the observable occupant quanta $\Psi_{\text{OQ}}(x)$ can be expressed as

$$S_{\text{OQ}}^{(3\text{br})}[\Psi_{\text{OQ}}, f^A] \stackrel{\text{def}}{=} \int_{\mathcal{WV}_{\text{sq}}} d^4x \sqrt{|\det(\gamma_{\mu\nu})|} \mathcal{L}_{\text{OQ}}^{(3\text{br})}(\Psi_{\text{OQ}}, \partial_\mu f^A, \dots), \quad (7.1)$$

where $\gamma_{\mu\nu} = \partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}}$. This action $S_{\text{OQ}}^{(3\text{br})}[\Psi_{\text{OQ}}, f^A]$ is assumed to be invariant under $ISO(1, D_{\text{amb}} - 1)$ and $\text{Diff}(4)$ like the 3-brane action $S_{\text{emb}}^{(3\text{br})}[f^A]$. Of course, the action $S_{\text{OQ}}^{(3\text{br})}$ in Eq. (7.1) may depend on a “bulk-field” $\Psi_{\text{bulk}}(Y^A)$ of the bulk spacetime $\mathbb{M}^{D_{\text{amb}}}$, whose field point Y^A should satisfy $Y^A = f^A(x^\mu)$. For example, when $\Psi_{\text{bulk}}(Y^A)$ is a bulk-tensor (e.g., a D_{amb} -dimensional vector), it can appear in the action $S_{\text{OQ}}^{(3\text{br})}$ through its pullback $(f^* \Psi_{\text{bulk}})(x^\mu)$ at sufficiently low energies.

Finally, the “**original**” universe action $S_{\text{univ}}^{(3\text{br})}[f^A, \Psi_{\text{OQ}}]$ at low energies is written as

$$S_{\text{univ}}^{(3\text{br})}[f^A, \Psi_{\text{OQ}}] = S_{\text{emb}}^{(3\text{br})}[f^A] + S_{\text{OQ}}^{(3\text{br})}[\Psi_{\text{OQ}}, f^A], \quad (7.2)$$

where the integral for the 3-brane action $S_{\text{emb}}^{(3\text{br})}[f^A]$ shares the same set \mathcal{WV}_{sq} with that for $S_{\text{OQ}}^{(3\text{br})}[\Psi_{\text{OQ}}, f^A]$ in Eq. (7.1).

For the original action $S_{\text{univ}}^{(3\text{br})}[f^A, \Psi_{\text{OQ}}]$ in Eq. (7.2), its **universe target** $\Sigma_{\text{target}}^{(\text{univ})} = \{f_{\text{sol}}^A\}$ is defined as

$$\Sigma_{\text{target}}^{(\text{univ})} \stackrel{\text{def}}{=} \{ f_{\text{sol}}^A : 4\text{DL embedding} \mid (\delta S_{\text{univ}}^{(3\text{br})}/\delta f^A)[f_{\text{sol}}^A, \Psi_{\text{OQ}}^{\text{sol}}] = 0 \} \subset \mathcal{F}_{\text{space}} , \quad (7.3)$$

where $(f_{\text{sol}}^A, \Psi_{\text{OQ}}^{\text{sol}})$ is a solution of the coupled Euler-Lagrange (E-L) equations

$$(\delta S_{\text{univ}}^{(3\text{br})}/\delta f^A)[f^A, \Psi_{\text{OQ}}] = 0 \quad \text{and} \quad (\delta S_{\text{univ}}^{(3\text{br})}/\delta \Psi_{\text{OQ}})[f^A, \Psi_{\text{OQ}}] = 0 . \quad (7.4)$$

Although the element f_{sol}^A of the set $\Sigma_{\text{target}}^{(\text{univ})} = \{f_{\text{sol}}^A\}$ satisfies the different equation (i.e., $\delta S_{\text{univ}}^{(3\text{br})}/\delta f^A = 0$) from $\delta S_{\text{emb}}^{(3\text{br})}/\delta f^A = 0$ for the target Σ_{target} in Eq. (5.1), the solution f_{sol}^A in Eq. (7.3) is still called a “4D-Lorentzian (4DL) solution.”

Since the universe target $\Sigma_{\text{target}}^{(\text{univ})}$ in Eq. (7.3) is defined similarly to the target Σ_{target} , we can similarly apply the AAT method in order to know the universe target $\Sigma_{\text{target}}^{(\text{univ})} = \{f_{\text{sol}}^A\}$, as follows: as in Sec. 5, the knowledge about the universe target $\Sigma_{\text{target}}^{(\text{univ})}$ is related to the **universe induced-metric set**

$$\Sigma_{\text{ind}}^{(\text{univ})} \stackrel{\text{def}}{=} \{ \gamma_{\mu\nu} \mid \gamma_{\mu\nu} = \partial_\mu f_{\text{sol}}^A \partial_\nu f_{\text{sol}}^B \eta_{AB}^{\text{bulk}} \text{ for every } f_{\text{sol}}^A \in \Sigma_{\text{target}}^{(\text{univ})} \} . \quad (7.5)$$

To study this universe induced-metric set $\Sigma_{\text{ind}}^{(\text{univ})} = \{\gamma_{\mu\nu}\}$ as in Sec. 5, we impose three requirements on the “**overlapping**” **universe action** $S_{\text{univ}}^{(\text{ovlp})} = \int_{\mathcal{S}_{\text{univ}}^{4\text{D}}} d^4x \hat{\mathcal{L}}_{\text{univ}}^{(\text{ovlp})}$:

- For the study of $\Sigma_{\text{ind}}^{(\text{univ})} = \{\gamma_{\mu\nu}\}$, the overlapping action $S_{\text{univ}}^{(\text{ovlp})}$ is a functional of the 4D Lorentzian metric $g_{\mu\nu}$ on the 4D manifold $\mathcal{S}_{\text{univ}}^{4\text{D}}$.
- The spacetime $\mathcal{S}_{\text{univ}}^{4\text{D}}$ for the action $S_{\text{univ}}^{(\text{ovlp})} = \int_{\mathcal{S}_{\text{univ}}^{4\text{D}}} d^4x \sqrt{|\det(g_{\mu\nu})|} \mathcal{L}_{\text{univ}}^{(\text{ovlp})}$ satisfies

$$\mathcal{S}_{\text{univ}}^{4\text{D}} = \mathcal{WV}_{\text{sq}} \quad (\text{cf. Eqs. (6.10) and (7.25)}) . \quad (7.6)$$

- The solution $g_{\mu\nu}^{\text{sol}\cdot\text{U}}$ (called the “U-metric”) of the equation $\delta S_{\text{univ}}^{(\text{ovlp})} = 0$ (see Eq. (7.18)) satisfies

$$g_{\mu\nu}^{\text{sol}\cdot\text{U}} = \gamma_{\mu\nu} \quad (\text{cf. Eqs. (6.9) and (7.25)}) . \quad (7.7)$$

Thus, since this induced metric $\gamma_{\mu\nu}$ depends on the observable occupant quanta through the 4DL solution f_{sol}^A due to Eqs. (7.3) and (7.5), it is natural to assume that the overlapping universe action $S_{\text{univ}}^{(\text{ovlp})}$ depends on these occupant quanta.

Therefore, we consider the overlapping universe action of the form

$$S_{\text{univ}}^{(\text{ovlp})}[g_{\mu\nu}, \psi_{\text{OQ}}] \stackrel{\text{def}}{=} S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}] + S_{\text{OQ}}^{(\text{ovlp})}[\psi_{\text{OQ}}, g_{\mu\nu}] , \quad (7.8)$$

where the “occupant-quantum (OQ) action”

$$S_{\text{OQ}}^{(\text{ovlp})}[\psi_{\text{OQ}}, g_{\mu\nu}] \stackrel{\text{def}}{=} \int_{\mathcal{S}_{\text{univ}}^{4\text{D}}} d^4x \sqrt{|\det(g_{\mu\nu})|} \mathcal{L}_{\text{OQ}}^{(\text{ovlp})}(\psi_{\text{OQ}}, g_{\mu\nu}, \dots), \quad (7.9)$$

and the metric action

$$S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}] \stackrel{\text{def}}{=} \int_{\mathcal{S}_{\text{univ}}^{4\text{D}}} d^4x \sqrt{|\det(g_{\mu\nu})|} \mathcal{L}_{\text{met}}^{(\text{ovlp})}(\Lambda_{\text{met}}; g_{\mu\nu}). \quad (7.10)$$

Because this metric action $S_{\text{met}}^{(\text{ovlp})}[g_{\mu\nu}]$ defined for $\mathcal{S}_{\text{univ}}^{4\text{D}}$ will be chosen to be invariant under $ISO(1, D_{\text{amb}} - 1)$ and $\text{Diff}(4)$ (see below), its Lagrangian $\mathcal{L}_{\text{met}}^{(\text{ovlp})}(\Lambda_{\text{met}}; g_{\mu\nu})$ in Eq. (7.10) has the same form as the Lagrangian in Eqs. (6.16) and (6.17)—we use the same notations.

Like $g_{\mu\nu}$ describing $\gamma_{\mu\nu}$ through $g_{\mu\nu}^{\text{sol}\cdot\text{U}} = \gamma_{\mu\nu}$, each “occupant-quantum (OQ) field” ψ_{OQ} in the overlapping action $S_{\text{univ}}^{(\text{ovlp})}[g_{\mu\nu}, \psi_{\text{OQ}}]$ describes its counterpart Ψ_{OQ} through

$$\psi_{\text{OQ}}^{\text{sol}} = \Psi_{\text{OQ}}^{\text{sol}} \quad (\text{see Eqs. (7.12) and (7.18)}), \quad (7.11)$$

where $\psi_{\text{OQ}}^{\text{sol}}$ is a part of the solution $(g_{\mu\nu}^{\text{sol}\cdot\text{U}}, \psi_{\text{OQ}}^{\text{sol}})$ of the coupled E-L equations

$$(\delta S_{\text{univ}}^{(\text{ovlp})}/\delta g_{\mu\nu})[g_{\mu\nu}, \psi_{\text{OQ}}] = 0 \quad \text{and} \quad (\delta S_{\text{univ}}^{(\text{ovlp})}/\delta \psi_{\text{OQ}})[g_{\mu\nu}, \psi_{\text{OQ}}] = 0. \quad (7.12)$$

When $\delta S_{\text{univ}}^{(\text{ovlp})}/\delta \psi_{\text{OQ}} = 0$ in Eq. (7.12) is compared with $\delta S_{\text{univ}}^{(3\text{br})}/\delta \Psi_{\text{OQ}} = 0$ in Eq. (7.4), we can find a simple method for achieving the above equality $\psi_{\text{OQ}}^{\text{sol}} = \Psi_{\text{OQ}}^{\text{sol}}$ under the assumption $g_{\mu\nu}^{\text{sol}\cdot\text{U}} = \gamma_{\mu\nu}$ in Eq. (7.7), as follows: the original OQ action $S_{\text{OQ}}^{(3\text{br})}[\Psi_{\text{OQ}}, f^A]$ in Eq. (7.1) can satisfy, at least at low enough energies,

$$S_{\text{OQ}}^{(3\text{br})}[\Psi_{\text{OQ}}, f^A] = \tilde{S}_{\text{OQ}}^{(\text{ovlp})}[\Psi_{\text{OQ}}, f^A] \stackrel{\text{def}}{=} S_{\text{OQ}}^{(\text{ovlp})}[\Psi_{\text{OQ}}, \partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}}], \quad (7.13)$$

which is obtained by the replacements (i) $\psi_{\text{OQ}} \Rightarrow \Psi_{\text{OQ}}$ and (ii) $g_{\mu\nu} \Rightarrow \partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}}$ in the overlapping OQ action $S_{\text{OQ}}^{(\text{ovlp})}[\psi_{\text{OQ}}, g_{\mu\nu}]$ in Eq. (7.9).

Due to Eq. (7.13), the solution $\Psi_{\text{OQ}}^{\text{sol}}$ of $(\delta S_{\text{OQ}}^{(3\text{br})}/\delta \Psi_{\text{OQ}})[\Psi_{\text{OQ}}, f_{\text{sol}}^A] = 0$ from Eq. (7.4) is also a solution of the replaced-equation from Eq. (7.12)

$$\delta S_{\text{OQ}}^{(\text{ovlp})}/\delta \psi_{\text{OQ}}|_{\text{repl}} \stackrel{\text{def}}{=} (\delta S_{\text{OQ}}^{(\text{ovlp})}/\delta \psi_{\text{OQ}})[\psi_{\text{OQ}}, \partial_\mu f_{\text{sol}}^A \partial_\nu f_{\text{sol}}^B \eta_{AB}^{\text{bulk}}] = 0, \quad (7.14)$$

which contains f_{sol}^A unlike other replaced-equations in Eqs. (5.14) and (7.21) due to the assumption $g_{\mu\nu}^{\text{sol}\cdot\text{U}} = \gamma_{\mu\nu}$. In this manner, the equality $\psi_{\text{OQ}}^{\text{sol}} = \Psi_{\text{OQ}}^{\text{sol}}$ in Eq. (7.11) is achieved.

For this equality $\psi_{\text{OQ}}^{\text{sol}} = \Psi_{\text{OQ}}^{\text{sol}}$, the OQ field ψ_{OQ} shares the same $ISO(1, D_{\text{amb}} - 1)$ and $\text{Diff}(4)$ symmetry properties with its corresponding brane-field Ψ_{OQ} . For example, the OQ field ψ_{OQ} is a $\text{Diff}(4)$ -tensor or $SO(1, 3)$ -spinor of the spacetime $\mathcal{S}_{\text{univ}}^{4\text{D}}$ like its counterpart Ψ_{OQ} . Of course, the equality $\psi_{\text{OQ}}^{\text{sol}} = \Psi_{\text{OQ}}^{\text{sol}}$ may have a limited validity like $g_{\mu\nu}^{\text{sol}} = \gamma_{\mu\nu}$ in Eq. (6.9), which is valid only for the B-T overlap $\Sigma_{\text{B}\cap\text{T}}^{(\text{ovlp})} (\subset \Sigma_{\text{target}})$.

Due to Eq. (7.13), the original universe action $S_{\text{univ}}^{(3\text{br})}[f^A, \Psi_{\text{OQ}}]$ in Eq. (7.2) can satisfy

$$S_{\text{univ}}^{(3\text{br})}[f^A, \Psi_{\text{OQ}}] = S_{\text{emb}}^{(3\text{br})}[f^A] + \tilde{S}_{\text{OQ}}^{(\text{ovlp})}[\Psi_{\text{OQ}}, f^A] \quad \text{at low enough energies.} \quad (7.15)$$

Hamilton's principle $\delta S_{\text{univ}}^{(3\text{br})}/\delta f^A = 0$ gives the equation of motion for the space 3-brane

$$\partial_\mu (\mathcal{T}_{3\text{br}} \sqrt{|\det(\gamma_{\rho\sigma})|} \gamma^{\mu\nu} \partial_\nu f^B \eta_{AB}^{\text{bulk}}) + \cdots = \partial_\mu (\sqrt{|\det(\gamma_{\rho\sigma})|} T_{\text{OQ}}^{\mu\nu} \partial_\nu f^B \eta_{AB}^{\text{bulk}}), \quad (7.16)$$

where

$$T_{\text{OQ}\mu\nu} \stackrel{\text{def}}{=} - \frac{2}{\sqrt{|\det(\gamma_{\alpha\beta})|}} \frac{\delta \tilde{S}_{\text{OQ}}^{(\text{ovlp})}}{\delta \gamma^{\mu\nu}}. \quad (7.17)$$

Since the OQ action $\tilde{S}_{\text{OQ}}^{(\text{ovlp})}[\Psi_{\text{OQ}}, f^A]$ is added to the 3-brane action $S_{\text{emb}}^{(3\text{br})}[f^A]$, the equation of motion in Eq. (7.16) is changed from Eq. (4.35).

As in Sec. 5, for the overlapping universe action $S_{\text{univ}}^{(\text{ovlp})}[g_{\mu\nu}, \psi_{\text{OQ}}]$ in Eq. (7.8), its **universe cartridge** $\Sigma_{\text{univ}}^{(\text{sol})} = \{g_{\mu\nu}^{\text{sol}\cdot\text{U}}\}$ is defined as

$$\Sigma_{\text{univ}}^{(\text{sol})} \stackrel{\text{def}}{=} \{g_{\mu\nu}^{\text{sol}\cdot\text{U}} \mid (\delta S_{\text{univ}}^{(\text{ovlp})}/\delta g_{\mu\nu})[g_{\mu\nu}^{\text{sol}\cdot\text{U}}, \psi_{\text{OQ}}^{\text{sol}}] = 0\}, \quad (7.18)$$

where $(g_{\mu\nu}^{\text{sol}\cdot\text{U}}, \psi_{\text{OQ}}^{\text{sol}})$ is the solution of the coupled E-L equations in Eq. (7.12).

Then, the **universe bullet** $\Sigma_{\text{bullet}}^{(\text{univ})} = \{f_{\text{bul}}^A\}$ of the overlapping action $S_{\text{univ}}^{(\text{ovlp})}[g_{\mu\nu}, \psi_{\text{OQ}}]$ is defined as

$$\Sigma_{\text{bullet}}^{(\text{univ})} \stackrel{\text{def}}{=} \{f^A \mid \partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}} = g_{\mu\nu}^{\text{sol}\cdot\text{U}} \text{ for every } g_{\mu\nu}^{\text{sol}\cdot\text{U}} \in \Sigma_{\text{univ}}^{(\text{sol})}\} \subset \mathcal{F}_{\text{space}}. \quad (7.19)$$

Like the universe target $\Sigma_{\text{target}}^{(\text{univ})} = \{f_{\text{sol}}^A\}$ in Eq. (7.3), the universe bullet $\Sigma_{\text{bullet}}^{(\text{univ})} = \{f_{\text{bul}}^A\}$ depends on the occupant quanta through the U-metric $g_{\mu\nu}^{\text{sol}\cdot\text{U}}$ in Eq. (7.19), because this solution metric $g_{\mu\nu}^{\text{sol}\cdot\text{U}}$ depends on the occupant quanta ψ_{OQ} through, e.g., the ψ_{OQ} -dependent equation $(\delta S_{\text{univ}}^{(\text{ovlp})}/\delta g_{\mu\nu})[g_{\mu\nu}, \psi_{\text{OQ}}] = 0$ in Eq. (7.12).

Because the action $S_{\text{univ}}^{(\text{ovlp})}[g_{\mu\nu}, \psi_{\text{OQ}}]$ in Eq. (7.8) is an *overlapping* one, the “universe B-T overlap” $\Sigma_{\text{B}\cap\text{T}}^{(\text{univ})} \stackrel{\text{def}}{=} \Sigma_{\text{bullet}}^{(\text{univ})} \cap \Sigma_{\text{target}}^{(\text{univ})}$ is not the empty set, i.e.,

$$\Sigma_{\text{B}\cap\text{T}}^{(\text{univ})} \neq \emptyset, \quad (7.20)$$

where $\Sigma_{\text{B}\cap\text{T}}^{(\text{univ})} = \{f_{\text{ove}}^A\}$ is assumed to contain a low-energy motion (e.g., $|\partial| \ll \Lambda_{\text{met}}$) which the space 3-brane can perform in the ambient spacetime $\mathbb{M}^{D_{\text{amb}}}$. As in Sec. 5, the universe B-T overlap $\Sigma_{\text{B}\cap\text{T}}^{(\text{univ})} = \{f_{\text{ove}}^A\}$ is the maximum knowledge which we can obtain about the universe target $\Sigma_{\text{target}}^{(\text{univ})} = \{f_{\text{sol}}^A\}$ by using the overlapping universe action $S_{\text{univ}}^{(\text{ovlp})}[g_{\mu\nu}, \psi_{\text{OQ}}]$.

Until now, we have presented the “two-step AAT method” for the overlapping universe action $S_{\text{univ}}^{(\text{ovlp})}[g_{\mu\nu}, \psi_{\text{OQ}}]$ (cf. Sec. 5):

- Step 1: finding a solution $g_{\mu\nu}^{\text{sol}\cdot\text{U}}$ of the coupled equations $(\delta S_{\text{univ}}^{(\text{ovlp})}/\delta g_{\mu\nu})[g_{\mu\nu}, \psi_{\text{oq}}] = 0$ and $(\delta S_{\text{univ}}^{(\text{ovlp})}/\delta \psi_{\text{oq}})[g_{\mu\nu}, \psi_{\text{oq}}] = 0$ in Eq. (7.12), and next
- Step 2: finding a solution f_{sol}^A of the new rifle PDE $\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}} = g_{\mu\nu}^{\text{sol}\cdot\text{U}}$.

Instead of this two-step AAT method, as in Sec. 5, we try another method of eliminating the metric $g_{\mu\nu}$ from those E-L equations $\delta S_{\text{univ}}^{(\text{ovlp})}/\delta g_{\mu\nu} = 0$ and $\delta S_{\text{univ}}^{(\text{ovlp})}/\delta \psi_{\text{oq}} = 0$ by inserting the PDE $\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}} = g_{\mu\nu}$ into them. Namely, we solve the coupled replaced-equations

$$\delta S_{\text{univ}}^{(\text{ovlp})}/\delta g_{\mu\nu}|_{\text{repl}} = 0 \quad \text{and} \quad \delta S_{\text{univ}}^{(\text{ovlp})}/\delta \psi_{\text{oq}}|_{\text{repl}} = 0 \quad (\text{cf. Eq. (5.14)}) , \quad (7.21)$$

where $\delta S_{\text{univ}}^{(\text{ovlp})}/\delta Z|_{\text{repl}} \stackrel{\text{def}}{=} (\delta S_{\text{univ}}^{(\text{ovlp})}/\delta Z)[\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}}, \psi_{\text{oq}}]$ for $Z = g_{\mu\nu}, \psi_{\text{oq}}$. The former replaced-equation $\delta S_{\text{univ}}^{(\text{ovlp})}/\delta g_{\mu\nu}|_{\text{repl}} = 0$ is expressed as $\delta \tilde{S}_{\text{univ}}^{(\text{ovlp})}/\delta(\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}}) = 0$, where $\tilde{S}_{\text{univ}}^{(\text{ovlp})}[f^A, \psi_{\text{oq}}] \stackrel{\text{def}}{=} S_{\text{univ}}^{(\text{ovlp})}[\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}}, \psi_{\text{oq}}]$ (cf. Eqs. (5.17) and (5.20)).

Since solving the coupled replaced-equations in Eq. (7.21) is the same as solving the new rifle PDE $\partial_\mu f^A \partial_\nu f^B \eta_{AB}^{\text{bulk}} = g_{\mu\nu}^{\text{sol}\cdot\text{U}}$ of the two-step AAT method, the solution set for Eq. (7.21)

$$\Sigma_{\text{univ}}^{(\text{sol})} \stackrel{\text{def}}{=} \{ f^A \mid \delta S_{\text{univ}}^{(\text{ovlp})}/\delta g_{\mu\nu}|_{\text{repl}} = 0 \quad \text{and} \quad \delta S_{\text{univ}}^{(\text{ovlp})}/\delta \psi_{\text{oq}}|_{\text{repl}} = 0 \} \quad (7.22)$$

is equal to the universe bullet $\Sigma_{\text{bullet}}^{(\text{univ})} = \{ f_{\text{bul}}^A \}$ in Eq. (7.19), namely,

$$\Sigma_{\text{univ}}^{(\text{sol})} = \Sigma_{\text{bullet}}^{(\text{univ})} \quad (\text{cf. Eq. (5.16)}) . \quad (7.23)$$

This equality $\Sigma_{\text{univ}}^{(\text{sol})} = \Sigma_{\text{bullet}}^{(\text{univ})}$ means that the universe bullet $\Sigma_{\text{bullet}}^{(\text{univ})} = \{ f_{\text{bul}}^A \}$ is also produced by the coupled replaced-equations $\delta S_{\text{univ}}^{(\text{ovlp})}/\delta g_{\mu\nu}|_{\text{repl}} = \delta S_{\text{univ}}^{(\text{ovlp})}/\delta \psi_{\text{oq}}|_{\text{repl}} = 0$.

Therefore, these replaced-equations $\delta S_{\text{univ}}^{(\text{ovlp})}/\delta g_{\mu\nu}|_{\text{repl}} = \delta S_{\text{univ}}^{(\text{ovlp})}/\delta \psi_{\text{oq}}|_{\text{repl}} = 0$ can be used instead of the E-L equations $\delta S_{\text{univ}}^{(3\text{br})}/\delta f^A = \delta S_{\text{univ}}^{(3\text{br})}/\delta \Psi_{\text{OQ}} = 0$ in Eq. (7.4) *within the universe B-T overlap* $\Sigma_{\text{B}\cap\text{T}}^{(\text{univ})}$ (see below Eq. (5.23)). In other words, within this B-T overlap $\Sigma_{\text{B}\cap\text{T}}^{(\text{univ})}$, the replaced-equations from $S_{\text{univ}}^{(\text{ovlp})}[g_{\mu\nu}, \psi_{\text{oq}}]$ are “equivalent” to the E-L equations from $S_{\text{univ}}^{(3\text{br})}[f^A, \Psi_{\text{OQ}}]$.

When a “single” overlapping action $S_{\text{univ}}^{(\text{ovlp})}[g_{\mu\nu}, \psi_{\text{oq}}]$ is discovered as a result of investigation, we can assume that the replaced-equations from this discovered action $S_{\text{univ}}^{(\text{ovlp})}[g_{\mu\nu}, \psi_{\text{oq}}]$ are applied, at least, to many and various motions which the space 3-brane can perform in the ambient spacetime $\mathbb{M}^{D_{\text{amb}}}$ at low enough energies. Namely, the AAT method using the single discovered action $S_{\text{univ}}^{(\text{ovlp})}[g_{\mu\nu}, \psi_{\text{oq}}]$ is valid for those many and various low-energy motions of the space 3-brane. (For a further study, see our next paper [14].) Of course, the discovered action $S_{\text{univ}}^{(\text{ovlp})}[g_{\mu\nu}, \psi_{\text{oq}}]$ can change, depending on observation energies.

Suppose a low-energy motion of the space 3-brane is described by a 4DL solution $f_{\text{sol}}^A(x^\mu)$. Then, each momentum p^μ in the Fourier transform of $f_{\text{sol}}^A(x^\mu)$ satisfies $|p^\mu| \ll \Lambda_{\text{cont}}$ for all μ . In this Fourier-transform context, the low-energy motion $f_{\text{sol}}^A(x^\mu)$ is expressed as

$$|\partial| \ll \Lambda_{\text{cont}} . \quad (7.24)$$

For this low-energy motion f_{sol}^A of $|\partial| \ll \Lambda_{\text{cont}}$, the U-metric $g_{\mu\nu}^{\text{sol}\cdot\text{U}}$ ($= \partial_\mu f_{\text{sol}}^A \partial_\nu f_{\text{sol}}^B \eta_{AB}^{\text{bulk}}$ by Eq. (7.7)) is also expressed as $|\partial| \ll \Lambda_{\text{cont}}$.

As in Sec. 6, we choose the “invariant case” that the overlapping action $S_{\text{univ}}^{(\text{ovlp})}[g_{\mu\nu}, \psi_{\text{oq}}]$ in Eq. (7.8) is invariant under $ISO(1, D_{\text{amb}} - 1)$ and $\text{Diff}(4)$ like the original one $S_{\text{univ}}^{(3\text{br})}[f^A, \Psi_{\text{OQ}}]$, irrespective of whether $S_{\text{univ}}^{(\text{ovlp})}[g_{\mu\nu}, \psi_{\text{oq}}]$ is an ME or PE action. Thus, the metric Lagrangian $\mathcal{L}_{\text{met}}^{(\text{ovlp})}(\Lambda_{\text{met}}; g_{\mu\nu})$ in $S_{\text{univ}}^{(\text{ovlp})}[g_{\mu\nu}, \psi_{\text{oq}}]$ shares the same form with that in Eqs. (6.16) and (6.17). Then, the $\text{Diff}(4)$ -invariant universe action $S_{\text{univ}}^{(\text{ovlp})}[g_{\mu\nu}, \psi_{\text{oq}}]$ can contain (i) the Einstein-Hilbert action and (ii) the action for matter (i.e., occupant quanta), both of which are the essential parts of General Relativity.

Within the region $|\partial| \ll \Lambda_{\text{cont}}$ in Eq. (7.24), we have the equality for the two spacetime manifolds (see Eqs. (7.6) and (7.7))

$$(\mathcal{S}_{\text{univ}}^{4\text{D}}, g_{\mu\nu}^{\text{sol}\cdot\text{U}}) = (\mathcal{WV}_{\text{sq}}, \gamma_{\mu\nu}) . \quad (7.25)$$

Note the 4D emergent spacetime $(\mathcal{WV}_{\text{sq}}, \gamma_{\mu\nu})$ is determined by the 4DL solution f_{sol}^A for the E-L equations $\delta S_{\text{univ}}^{(3\text{br})}/\delta f^A = \delta S_{\text{univ}}^{(3\text{br})}/\delta \Psi_{\text{OQ}} = 0$ in Eq. (7.3). This emergent manifold $(\mathcal{WV}_{\text{sq}}, \gamma_{\mu\nu})$ occupying $\mathbb{M}^{D_{\text{amb}}}$ is the exact or true spacetime which provides exact values for our spacetime measurements (see below Eq. (6.11)).

As said below Eq. (7.10), since the metric Lagrangian $\mathcal{L}_{\text{met}}^{(\text{ovlp})}(\Lambda_{\text{met}}; g_{\mu\nu})$ in Eq. (7.10) shares the same form with that in Eqs. (6.16) and (6.17), we use the same notations. Due to the power-law behaviors $(\partial/\Lambda_{\text{met}})^{d_k}$ in Eqs. (6.16) and (6.17), the most dominant term in $\mathcal{L}_{\text{met}}^{(\text{ovlp})}(\Lambda_{\text{met}}; g_{\mu\nu})$ for $|\partial| \ll \Lambda_{\text{met}}$ is the $d_k = 0$ Lagrangian $\mathcal{L}_{\text{met}}^{(0)} \stackrel{\text{def}}{=} c_0 \Lambda_{\text{met}}^4$, which contributes to a cosmological constant. Moreover, the next dominant term is the $d_k = 2$ Lagrangian $\mathcal{L}_{\text{met}}^{(2)} \stackrel{\text{def}}{=} c_2 \Lambda_{\text{met}}^2 R$, which contains only the two-derivative terms of the metric $g_{\mu\nu}$.

As assumed before, the AAT method using the overlapping universe action $S_{\text{univ}}^{(\text{ovlp})}[g_{\mu\nu}, \psi_{\text{oq}}]$ is applied to various low-energy motions (i.e., $|\partial| \ll \Lambda_{\text{cont}}$) of the space 3-brane. Within the region $|\partial| \ll \Lambda_{\text{cont}}$, we can find a low-energy region $|\partial| \ll \Lambda_{\text{met}}$ ($\lesssim O(\Lambda_{\text{cont}})$) in which the overlapping action $S_{\text{univ}}^{(\text{ovlp})}[g_{\mu\nu}, \psi_{\text{oq}}]$ has the approximation

$$S_{\text{univ}}^{(\text{ovlp})}[g_{\mu\nu}, \psi_{\text{oq}}] \approx S_{\text{univ}}^{(\leq 2)}[g_{\mu\nu}, \psi_{\text{oq}}^{\text{low}}] , \quad (7.26)$$

where

$$S_{\text{univ}}^{(\leq 2)}[g_{\mu\nu}, \psi_{\text{oq}}^{\text{low}}] \stackrel{\text{def}}{=} S_{\text{met}}^{(\leq 2)}[g_{\mu\nu}] + S_{\text{OQ}}^{(\text{low})}[\psi_{\text{oq}}^{\text{low}}, g_{\mu\nu}] , \quad (7.27)$$

$$S_{\text{met}}^{(\leq 2)}[g_{\mu\nu}] \stackrel{\text{def}}{=} \int_{\mathcal{S}_{\text{univ}}^{4\text{D}(\leq 2)}} d^4x \sqrt{|\det(g_{\mu\nu})|} (c_0 \Lambda_{\text{met}}^4 + c_2 \Lambda_{\text{met}}^2 R) . \quad (7.28)$$

In Eq. (7.27), the new OQ action $S_{\text{OQ}}^{(\text{low})}[\psi_{\text{oq}}^{\text{low}}, g_{\mu\nu}]$ is the low-energy approximation of its full theory $S_{\text{OQ}}^{(\text{ovlp})}[\psi_{\text{oq}}, g_{\mu\nu}]$ in Eq. (7.9). Namely, $S_{\text{OQ}}^{(\text{low})}[\psi_{\text{oq}}^{\text{low}}, g_{\mu\nu}]$ contains only the low-dimension interactions of $S_{\text{OQ}}^{(\text{ovlp})}[\psi_{\text{oq}}, g_{\mu\nu}]$ which are not negligible in the low-energy region

$|\partial| \ll \Lambda_{\text{met}}$. Of course, some heavy OQ fields (say, $\psi_{\text{oq}}^{\text{heavy}}$) appearing in $S_{\text{OQ}}^{(\text{ovlp})}[\psi_{\text{oq}}, g_{\mu\nu}]$ may be decoupled from its low-energy approximation $S_{\text{OQ}}^{(\text{low})}[\psi_{\text{oq}}^{\text{low}}, g_{\mu\nu}]$. In Eq. (7.28), the integral for $S_{\text{met}}^{(\leq 2)}[g_{\mu\nu}]$ undergoes the replacement $\mathcal{S}_{\text{univ}}^{4\text{D}} \Rightarrow \mathcal{S}_{\text{univ}}^{4\text{D}(\leq 2)}$, which is also undergone by the integral for $S_{\text{OQ}}^{(\text{low})}[\psi_{\text{oq}}^{\text{low}}, g_{\mu\nu}]$.

Due to the approximate equality $S_{\text{univ}}^{(\text{ovlp})} \approx S_{\text{univ}}^{(\leq 2)}$ in Eq. (7.26), the solution $(g_{\mu\nu}^{\text{sol}\cdot\text{U}}, \psi_{\text{oq}}^{\text{sol}})$ of the “exact” equations $\delta S_{\text{univ}}^{(\text{ovlp})}/\delta g_{\mu\nu} = 0$ and $\delta S_{\text{univ}}^{(\text{ovlp})}/\delta \psi_{\text{oq}} = 0$ in Eq. (7.12) has the approximate equalities

$$g_{\mu\nu}^{\text{sol}\cdot\text{U}} \approx g_{\mu\nu}^{\text{sol}(\leq 2)} \quad \text{and} \quad \psi_{\text{oq}}^{\text{sol}} \approx \psi_{\text{oq}}^{\text{sol}(\leq 2)}, \quad (7.29)$$

where $(g_{\mu\nu}^{\text{sol}(\leq 2)}, \psi_{\text{oq}}^{\text{sol}(\leq 2)})$ is the solution of the “approximate” equations $\delta S_{\text{univ}}^{(\leq 2)}/\delta g_{\mu\nu} = 0$ and $\delta S_{\text{univ}}^{(\leq 2)}/\delta \psi_{\text{oq}} = 0$. In addition, $g_{\mu\nu}^{\text{sol}\cdot\text{U}} \approx g_{\mu\nu}^{\text{sol}(\leq 2)}$ in Eq. (7.29) implies the approximate equality for the spacetime

$$\mathcal{S}_{\text{univ}}^{4\text{D}} \approx \mathcal{S}_{\text{univ}}^{4\text{D}(\leq 2)}. \quad (7.30)$$

In the low-energy region $|\partial| \ll \Lambda_{\text{met}}$, the approximate equation $\delta S_{\text{univ}}^{(\leq 2)}/\delta g_{\mu\nu} = 0$ is the same as Einstein’s equation with the three parameters c_0 , c_2 and Λ_{met}

$$G_{\mu\nu} - \frac{c_0 \Lambda_{\text{met}}^2}{2c_2} g_{\mu\nu} = \frac{1}{2c_2 \Lambda_{\text{met}}^2} T_{\mu\nu}^{(\text{low})}, \quad (7.31)$$

where $1/(2c_2 \Lambda_{\text{met}}^2)$ corresponds to $8\pi G_{\text{N}}$ of the ordinary Einstein’s equation, and

$$T_{\mu\nu}^{(\text{low})} \stackrel{\text{def}}{=} - \frac{2}{\sqrt{|\det(g_{\alpha\beta})|}} \frac{\delta S_{\text{OQ}}^{(\text{low})}}{\delta g^{\mu\nu}}. \quad (7.32)$$

For $\gamma_{\mu\nu} = g_{\mu\nu}$, the tensor $T_{\text{OQ}\mu\nu}$ in Eq. (7.17) satisfies $T_{\text{OQ}\mu\nu} \approx T_{\mu\nu}^{(\text{low})}$ at the low energies $|\partial| \ll \Lambda_{\text{met}}$.

Suppose that the “scalar $\times g_{\mu\nu}$ ” term in Eq. (7.31) is negligible as in the observed ΛCDM model [13]. Then, a spherical massive object can produce the Schwarzschild metric $g_{\mu\nu}^{(\text{S})}$, which leads to the gravitational potential $\phi_{\text{grav}} = -(g_{00}^{(\text{S})} + 1)/2$ in the Newtonian limit [1, 2, 3]. Since this potential $\phi_{\text{grav}} (\propto 1/c_2 \Lambda_{\text{met}}^2)$ depends on $c_2 \Lambda_{\text{met}}^2$ strongly, the value of $c_2 \Lambda_{\text{met}}^2$ can be easily determined by the comparison with observed data.

In Eq. (7.28), when the coefficient $c_2 \Lambda_{\text{met}}^2$ in the integrand satisfies

$$c_2 \Lambda_{\text{met}}^2 = 1/16\pi G_{\text{N}} \quad (\text{i.e., } \Lambda_{\text{met}} = M_{\text{P}}/\sqrt{16\pi c_2}), \quad (7.33)$$

the approximate metric action $S_{\text{met}}^{(\leq 2)}[g_{\mu\nu}]$ may not be distinguished from the Einstein-Hilbert action $S_{\text{EH}}^{(\text{DE})}[\mathbf{g}_{\mu\nu}]$ with a dark energy (DE) density ρ_{DE}

$$S_{\text{EH}}^{(\text{DE})}[\mathbf{g}_{\mu\nu}] \stackrel{\text{def}}{=} \int_{\mathcal{S}_{\text{GR}}} d^4x \sqrt{|\det(\mathbf{g}_{\mu\nu})|} (\mathbf{R}/16\pi G_{\text{N}} - \rho_{\text{DE}}), \quad (7.34)$$

where the Ricci scalar \mathbf{R} of General Relativity (GR) is built from the GR metric $\mathbf{g}_{\mu\nu}$.

To be more concrete, we consider in what situation General Relativity is valid: since using General Relativity of the $(\partial/M_P)^{d_k \leq 2}$ terms means neglecting all the higher-order $(\partial/M_P)^{d_k \geq 4}$ terms of the Lagrangian $\mathcal{L}_{\text{met}}^{(\text{ovlp})}(\Lambda_{\text{met}}; g_{\mu\nu})$ in Eq. (7.10), it is important to estimate the size of ∂ . As a measure of $|\partial|$, Kretschmann scalar $K \stackrel{\text{def}}{=} \mathbf{R}^{\mu\nu\rho\sigma} \mathbf{R}_{\mu\nu\rho\sigma}$ is used due to $K = O(\partial^4)$.

In this context, we deal with an extremely strong gravity related to a Schwarzschild black hole of mass M_{bh} , whose Kretschmann scalar is $K|_{\text{at } r} = 48M_{\text{bh}}^2/M_P^4 r^6$ [3]. Outside the Schwarzschild radius $R_S = 2M_{\text{bh}}/M_P^2$ (i.e., $r > R_S$), the scalar satisfies $K|_{\text{at } r} < K|_{\text{at } R_S}$, which leads to $|\partial|/M_P \lesssim M_P/M_{\text{bh}}$ due to $K|_{\text{at } r} = O(\partial^4)$ and $K|_{\text{at } R_S} = O(M_P/M_{\text{bh}})^4 M_P^4$. Then, for $M_{\text{bh}} \gtrsim M_\odot$ ($\approx 10^{38} M_P$), the result $|\partial|/M_P \ll 1$ implies that General Relativity is valid outside the event horizon at $r = R_S$.

Meanwhile, inside this event horizon, there is a radius R_∞ satisfying $K|_{\text{at } R_\infty} = O(M_P^4)$, which produces $R_\infty = O(M_{\text{bh}}/M_P)^{1/3} M_P^{-1}$ ($M_P^{-1} \ll R_\infty \ll R_S$). For $r \lesssim R_\infty$, $|\partial|/M_P \gtrsim 1$ (i.e., $d_{\text{max}} \rightarrow \infty$) implies that General Relativity is not valid far inside the event horizon. Similarly, for $r \ll R_{\text{cont}}$ with $K|_{\text{at } R_{\text{cont}}} = O(\Lambda_{\text{cont}})^4$, $|\partial|/\Lambda_{\text{cont}} \gg 1$ implies that the continuum approximation of the quasi-3D object breaks down—the above black hole may not have the singularity at its center $r = 0$.

To sum up, in the low-energy region $|\partial| \ll \Lambda_{\text{met}}$, General Relativity can be a good approximation of the overlapping universe action $S_{\text{univ}}^{(\text{ovlp})}[g_{\mu\nu}, \psi_{\text{oq}}]$, which is an essential part of the AAT method for studying the original universe action $S_{\text{univ}}^{(3\text{br})}[f^A, \Psi_{\text{OQ}}]$ (i.e., the principle governing the motions of the space 3-brane). Note that our spacetime $\mathcal{WV}_{\text{sq}} (\approx \mathcal{S}_{\text{GR}})$ can have its own gravity (i.e., General Relativity) although the ambient spacetime $\mathcal{S}^{D_{\text{amb}}}$ does not have any “bulk gravity” (i.e., $\mathcal{S}^{D_{\text{amb}}} = \mathbb{M}^{D_{\text{amb}}}$).

In the case of

$$S_{\text{met}}^{(\leq 2)}[\mathbf{g}_{\mu\nu}] = S_{\text{EH}}^{(\text{DE})}[\mathbf{g}_{\mu\nu}] , \quad (7.35)$$

the solution metric $\mathbf{g}_{\mu\nu}^{\text{sol}}$ and the spacetime \mathcal{S}_{GR} of General Relativity have the equalities

$$\mathbf{g}_{\mu\nu}^{\text{sol}} = g_{\mu\nu}^{\text{sol}(\leq 2)} (\approx g_{\mu\nu}^{\text{sol}\cdot\text{U}} = \gamma_{\mu\nu} \text{ due to Eqs. (7.25) and (7.29)}) , \quad (7.36)$$

$$\mathcal{S}_{\text{GR}} = \mathcal{S}_{\text{univ}}^{4\text{D}(\leq 2)} (\approx \mathcal{S}_{\text{univ}}^{4\text{D}} = \mathcal{WV}_{\text{sq}} \text{ due to Eqs. (7.25) and (7.30)}) . \quad (7.37)$$

According to Eqs. (7.36) and (7.37), the spacetime $(\mathcal{S}_{\text{GR}}, \mathbf{g}_{\mu\nu}^{\text{sol}})$ of General Relativity can be at least a good approximation of the exact or true spacetime $(\mathcal{WV}_{\text{sq}}, \gamma_{\mu\nu})$, which is formed by many space quanta occupying the ambient spacetime $\mathbb{M}^{D_{\text{amb}}}$. This supports the space-quantum hypothesis in Eq. (2.9). If the exact equality $S_{\text{univ}}^{(\text{ovlp})} = S_{\text{univ}}^{(\leq 2)}$ really happens instead of the approximate one in Eq. (7.26), the “approximate equality” signs \approx in Eqs. (7.29), (7.30), (7.36) and (7.37) are replaced with the equality signs $=$.

Finally, until now, we have considered only the special situation that the ambient spacetime $\mathcal{S}^{D_{\text{amb}}}$ is the flat manifold $\mathbb{M}^{D_{\text{amb}}} = (\mathbb{R}^{D_{\text{amb}}}, \eta_{AB}^{\text{bulk}})$, in which the inertial bulk observer O_{bulk} uses the inertial bulk-coordinates Y^A (see Sec. 3).

However, the ambient spacetime $\mathcal{S}^{D_{\text{amb}}}$ can be a *curved* manifold having a general bulk

metric g_{AB}^{bulk} , implying the replacements

$$\mathbb{M}^{D_{\text{amb}}} \Rightarrow \mathcal{S}^{D_{\text{amb}}} , \quad (7.38)$$

$$\eta_{AB}^{\text{bulk}} \Rightarrow g_{AB}^{\text{bulk}} , \quad (7.39)$$

$$ISO(1, D_{\text{amb}} - 1) \Rightarrow \text{Diff}(D_{\text{amb}}) . \quad (7.40)$$

For these replacements, our previous studies can be extended similarly.

The topology of the ambient spacetime $\mathcal{S}^{D_{\text{amb}}}$ may be, for example, $\mathbb{R}^{D_{\text{amb}}}$ or $\mathbb{R}^4 \times \mathbb{T}^{D_{\text{amb}}-4}$, where $\mathbb{T}^{D_{\text{amb}}-4}$ is a $(D_{\text{amb}} - 4)$ -dimensional spacelike torus. For $D_{\text{amb}} = 4$, when the topology of $\mathcal{S}^{D_{\text{amb}}}$ is \mathbb{R}^4 , the topology of the space 3-brane can be \mathbb{R}^3 , implying the world volume \mathcal{WV}_{sq} of this space 3-brane may be spatially flat. Then, due to $\mathcal{S}_{\text{GR}} \approx \mathcal{WV}_{\text{sq}}$ in Eq. (7.37), the corresponding spacetime \mathcal{S}_{GR} of General Relativity may be spatially flat, which can agree with the observed Λ CDM model [13]. For $D_{\text{amb}} \geq 5$, the same conclusions can be reached even for the topology $\mathbb{R}^4 \times \mathbb{T}^{D_{\text{amb}}-4}$ of $\mathcal{S}^{D_{\text{amb}}}$, when the size of this torus $\mathbb{T}^{D_{\text{amb}}-4}$ is much smaller than the distance d_{sq} between space quanta—at low energies $\ll \Lambda_{\text{cont}}$, the bulk spacetime $\mathcal{S}^{D_{\text{amb}}}$ can be observed as if its topology were \mathbb{R}^4 .

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